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Research Article

A family of Newton-Chebyshev type methods to find simple roots of nonlinear equations and their dynamics

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Abstract

In this work, a new family of Newton-Chebyshev type methods for solving nonlinear equations is presented. The dynamics of the Newton-Chebyshev family for the class of quadratic polynomials is analyzed and the convergence is established. We find the fixed and critical points. The stable and unstable behaviors are studied. The parameter space associated with the family is studied and finally, some dynamical planes that show different aspects of the dynamics of this family are presented.

Keywords: Nonlinear equations, Newton's method, Chebyshev's method, order of convergence, dynamic, quadratic polynomials, Newton-Chebyshev family.

1 Introduction

Iterative methods are usually necessary for solving nonlinear equations $f(x) = 0$, with $f : \mathbb{C} \to \mathbb{C}$. Several good methods exist in the literature: Newton, Halley and Chebyshev methods among others, see ([1]-[3]). The study of the dynamics of various methods was also done, see [4] for example.
In this paper, we analyze the dynamics of a new family of Newton-Chebyshev type methods for solving nonlinear equations when applied on quadratic polynomials.

1.1 Basic preliminaries

We now recall some preliminaries of complex dynamics (see [4], [13] and [51]) that we use in this work.
Given a rational function $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, where $\widehat{\mathbb{C}}$ is the Riemann sphere

Definition 1.1. *For $z \in \widehat{\mathbb{C}}$ we define its orbit as the set $orb(z) = \{z, R(z), R^2(z), \cdots, R^n(z), \cdots\}$.*

Definition 1.2. *A point z_0 is a fixed point of R if $R(z_0) = z_0$.*

Definition 1.3. *A critical point z_{cr} is a point such that $R'(z_{cr}) = 0$.*

Definition 1.4. *A fixed point z_0 is called attractor if $|R'(z_0)| < 1$, repulsive if $|R'(z_0)| > 1$, and parabolic or neutral if $|R'(z_0)| = 1$. If $|R'(z_0)| = 0$ then the fixed point is called superattractor. A superattractor fixed point is also a critical point.*

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Definition 1.5. *A fixed point z_0 that is not associated to the roots of the function $f(z)$ is called strange fixed point.*

Definition 1.6. *The basin of attraction of a attractor $\alpha \in \widehat{\mathbb{C}}$ is defined as the set of starting points whose orbits tend to α.*

This paper is organized as follows. In section 2, the Newton-Chebyshev family and its convergence are presented. In section 3, the dynamical behavior is analyzed and in section 4 final remarks are shown.

2 A family of Newton-Chebyshev type methods

In this section, we present the new family and its convergence. We recall that a sequence $\{x_n\}_{n\geq 0}$ converges to r with order of convergence p if there exists a $K(r) > 0$ such that

$$\lim_{n\to\infty} \frac{|x_{n+1} - r|}{|x_n - r|^p} = K(r)$$

and the error equation is

$$e_{n+1} = K(r)e_n^p + O(e_n^{p+1})$$

where $e_n = x_n - r$ and $K(r)$ is the asymptotic constant error.
In 1993 Hernández and Salanova [52] developed a family of Chebyshev-Halley type methods. Here, we present an uni-parametric family that allows us to study the evolution of the dynamics of Newton-Chebyshev family given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left(1 + AL_f(x_n)\right); \quad n = 0,1,2,\cdots \quad \textit{where} \quad L_f(z) = \frac{f(z)f''(z)}{\left(f'(z)\right)^2} \tag{2.1}$$

and where parameter A is complex and L_f is the degree of the logarithmic convexity function (see [53]-[55]). This family includes Newton's method for $A = 0$ and Chebyshev's method for $A = \frac{1}{2}$.
To begin the study of this family, we present the convergence in the following theorem.

Theorem 2.1. *Let $\alpha \in B$ be a simple root of a sufficiently differentiable function $f : B \to \Re$ for an open interval B. If x_0 is sufficiently close to α, then the family Newton-Halley type methods defined by (2.1) has at least second-order convergence, and satisfies the error equation:*

$$e_{n+1} = (1-2A)B_2e_n^2 + 2\left((4A-1)B_2^2 + (1-3A)B_3\right)e_n^3 + O\left(e_n^4\right) \tag{2.2}$$

where $e_n = x_n - \alpha$ is the error in the nth iterate and $B_j = \frac{f^{(j)}(\alpha)}{j!}$, $j = 1,2,\cdots$.

Proof. By Taylor series expansion around the simple root α in the nth iteration, we have

$$\begin{aligned} f(x_n) &= e_n + B_2e_n^2 + B_3e_n^3 + O\left(e_n^4\right) \\ f'(x_n) &= 1 + 2B_2e_n + 3B_3e_n^2 + 4B_4e_n^3 + O\left(e_n^4\right) \\ f''(x_n) &= 2B_2e_n + 6B_3e_n + 12B_4e_n^2 + 20B_5e_n^3 + O\left(e_n^4\right) \end{aligned}$$

Furthermore, it can be easily found that substituting in the terms involved is (2.1) we obtain

$$\begin{aligned} \frac{f(x_n)}{f'(x_n)} &= e_n - B_2e_n^2 + (2B_2^2 - 2B_3)e_n^3 + O\left(e_n^4\right) \\ L_f(x_n) &= 2B_2e_n + 6(B_3 - B_2^2)e_n^2 + 4(4B_2^3 - 7B_2B_3 + 3B_4)e_n^3 + O\left(e_n^4\right) \end{aligned}$$

which gives (2.2). This proves the theorem ☐

So, the family has order of convergence two except for Chebyshev's method which has order three.

3 Dynamical of the Newton-Chebyshev family

Here the author establishes the conjugacy class, also the fixed and critical points of this family in terms of the parameter A are calculated. Then the study of the fixed points, critical points and parameter space are presented. To finish this section several dynamical planes for different values of A selected from the parameter space are shown.

3.1 Conjugacy classes

In what remains of this paper we study the dynamics of the rational map R arising from Newton-Halley family (2.1)

$$R_f = z - \frac{f(z)}{f'(z)}\left(1 + AL_f(z)\right); \quad where \quad L_f(z) = \frac{f(z)f''(z)}{(f'(z))^2} \tag{3.3}$$

applied to $P_2(z) = a_2(z-z_1)(z-z_2)$. Let us first remember the following definition.

Definition 3.1. *[56]. Let f and g be two maps from the Riemann sphere into itself. An analytic conjugacy between f and g is an analytic diffeomorphism h from the Riemann sphere onto itself such that $h \circ f = g \circ h$.*

R_f has the following property for an analytic function f

Theorem 3.1. *(The Scaling Theorem). Let $f(z)$ be an analytical function on the Riemann sphere, and let $T(z) = \alpha z + \beta$, $\alpha \neq 0$, be an affine map. If $g(z) = f \circ T(z)$, then $T \circ R_g \circ T^{-1} = R_f(z)$. That is, R_f is analytically conjugate to R_g by T.*

Proof. With the iteration function $R(z)$, we have

$$R_g(T^{-1}(z)) = T^{-1}(z) - \frac{g\left(T^{-1}(z)\right)}{g'\left(T^{-1}(z)\right)}\left(1 + AL_g\left(T^{-1}(z)\right)\right) \quad with \quad L_g\left(T^{-1}(z)\right) = \frac{g\left(T^{-1}(z)\right)g''\left(T^{-1}(z)\right)}{\left(g'\left(T^{-1}(z)\right)\right)^2}$$

Since $\alpha T^{-1}(z) + \beta = z$, $g \circ T^{-1}(z) = f(z)$ and $\left(g \circ T^{-1}\right)'(z) = \frac{1}{\alpha} g'\left(T^{-1}(z)\right)$, we get $g'\left(T^{-1}(z)\right) = \alpha\left(g \circ T^{-1}\right)'(z) = \alpha f'(z)$, $g''\left(T^{-1}(z)\right) = \alpha^2 f''(z)$. We therefore have

$$\begin{aligned} T \circ R_g \circ T^{-1}(z) &= T\left(R_g(T^{-1}(z))\right) = \alpha R_g\left(T^{-1}(z)\right) + \beta \\ &= \alpha T^{-1}(z) - \frac{\alpha g\left(T^{-1}(z)\right)}{g'\left(T^{-1}(z)\right)}\left(1 + A\frac{g\left(T^{-1}(z)\right)g''\left(T^{-1}(z)\right)}{\left(g'\left(T^{-1}(z)\right)\right)^2}\right) + \beta = z - \frac{f(z)}{f'(z)}\left(1 + AL_f(z)\right) = R_f(z) \end{aligned}$$

□

Theorem 3.1 allows the study of the dynamics of the iteration function of Newton-Chebyshev family (2.1) for the polynomial $P_2(z) = a_2(z-z_1)(z-z_2)$ by means of the study of the polynomial $p(z) = (z-a)(z-b)$ where $a \neq b$.

Definition 3.2. *[10]. We say that a one-point iterative root-finding algorithm $p \to T_p$ has a universal Julia set (for polynomials of degree d) if there exists a rational map S such that for every degree d polynomial p, $J(T_p)$ is conjugate by a Möbius transformation to $J(S)$*

The following theorem establishes a universal Julia set for quadratics for our method (2.1).

Theorem 3.2. *For a rational map $R_p(z)$ arising from the method (2.1) applied to $p(z) = (z-a)(z-b), a \neq b$, $R_p(z)$ is conjugate via the Möbius transformation given by $M(z) = \frac{z-a}{z-b}$ to*

$$S(z) = \frac{z^2(z^2 + 2z + 1 - 2A)}{(1-2A)z^2 + 2z + 1} \tag{3.4}$$

Proof. Let $p(z) = (z-a)(z-b), a \neq b$ and $M(z) = \frac{z-a}{z-b}$ with $M^{-1}(z) = \frac{bz-a}{z-1}$. We then have

$$M \circ R_p \circ M^{-1}(z) = M\left(R_p\left(\frac{bz-a}{z-1}\right)\right) = \frac{z^2(z^2+2z+1-2A)}{(1-2A)z^2+2z+1}$$

□

We observe that parameters a and b do not appear in $S(z)$ because the Newton-Halley family complies with theorem 3.1.
The next subsections consider four specific values of A

3.1.*1* $A = 0$: ***Newton's Method***

In this case $S(z) = z^2$ and the fixed points are $z = 0$, $z = 1$ and $z = \infty$. As $S'(z) = 2z$ then $|S'(0)| = 0$, $|S'(1)| = 2$ and $|S'(\infty)| = \infty$. So, $z = 0$ and $z = \infty$ are superattractive fixed points. $z = 1$ is a repulsive strange fixed point.

3.1.*2* $A = \frac{1}{2}$: ***Chebyshev's Method***

If $A = \frac{1}{2}$ then $S(z) = \frac{(z+2)z^3}{2z+1}$ and $S'(z) = \frac{6z^2(z+1)^2}{(2z+1)^2}$. Superattractive fixed points are $z = 0$ and $z = \infty$ and exist three strange fixed points

$$\begin{aligned} z_1 &= 1 \\ z_{2,3} &= -\frac{3}{2} \pm \frac{\sqrt{5}}{2} \end{aligned}$$

Evaluating the derivative of S in the strange fixed points, then $|S'(1)| = \frac{8}{3}$ and $\left|S'\left(-\frac{3}{2} \pm \frac{\sqrt{5}}{2}\right)\right| = 6$ are obtained. So, the three strange fixed points are repulsive.

3.1.*3* ***An interesting case:*** $A = 2$

When $A = 2$ then $S(z) = -\frac{z^2(z+3)}{3z+1}$ and $S'(z) = -\frac{6z(z+1)^2}{(3z+1)^2}$. Superattractive fixed points are only $z = 0$ and $z = \infty$. And the strange fixed points are

$$z_{1,2} = -3 \pm 2\sqrt{2}$$

then $|S'(z_{1,2})| = 3$, whereby these are repulsive fixed points.

3.1.*4* ***Another interesting case:*** $A = -2$

When $A = -2$ then $S(z) = \frac{z^2\left(z^2+2z+5\right)}{5z^2+2z+1}$ and $S'(z) = \frac{2z(z+1)\left(5z^2-2z+5\right)}{\left(5z^2+2z+1\right)^2}$. Superattractive fixed points are only $z = 0$ and $z = \infty$. And the strange fixed point is $z = 1$ with multiplicity three. As $|S'(1)| = 1$, then this is a fixed point parabolic.

3.2 Study of the fixed points

The fixed points of S, for S defined in (3.4), are $z = 0$, $z = \infty$ and

$$\begin{aligned} z_1 &= 1 \\ z_{2,3} &= -(1+A) \pm \sqrt{A(A+2)} \end{aligned}$$

which are three roots of $(z-1)(z^2+2(1+A)z+1) = 0$.
To study the stability of the fixed points, we calculate $S'(z)$, so

$$S'(z) = \frac{2z(z+1)^2\left[(1-2A)z^2+2(1+A)z+1-2A\right]}{\left[(1-2A)z^2+2z+1\right]^2} \tag{3.5}$$

It is obvious from (3.5) that $z=0$ and $z=\infty$ are superattractive fixed points. The study of the stability of the other fixed points is now presented.
The operator $S'(z)$ given in (3.5) in $z=1$ gives

$$|S'(1)| = \left|\frac{4}{A-2}\right|, \quad (A \neq 2) \tag{3.6}$$

If we analyze this function, we obtain a horizontal asymptote in $|S'(1)|=0$ when $A \to \pm\infty$, and a vertical asymptote in $A=2$ (see section *3.1*.4).
In the following result we present the stability of the fixed point $z=1$.

Theorem 3.3. *The strange fixed point $z=1$ satisfies the following statements:*

1. *If $|A-2|>4$, then $z=1$ is an attractor*
2. *If $|A-2|=4$, then $z=1$ is a parabolic fixed point*
3. *If $A \neq 2$ and $|A-2|<4$, then $z=1$ is a repulsive fixed point.*

Proof. By simple inspection of (3.6). □

The operator $S'(z)$ in $z_{2,3}$ gives

$$\left|S'(z_{2,3})\right| = \frac{2\left|(A+1)\left(1+A\mp\sqrt{A(A+2)}\right)^2\left(A\mp\sqrt{A(A+2)}\right)^2\right|}{\left|A\left(A(2A+3)\mp(2A+1)\sqrt{A(A+2)}\right)^2\right|} = 2\left|\frac{A+1}{A}\right|, \quad (A\neq 0) \tag{3.7}$$

If we analyze this function, we obtain a horizontal asymptote in $|S'(z_{2,3})|=2$ when $A\to\infty$, and a vertical asymptote in $A=0$ (Newton's method).
In the following result, we present the stability of the fixed points $z=z_{2,3}$

Theorem 3.4. *The strange fixed points $z=z_{2,3}$ satisfy the following statements:*

1. *If $\left|A+\frac{4}{3}\right|<\frac{2}{3}$, then $z=z_{2,3}$ are attractors and, in particular, these are superattractors for $A=-1$*
2. *If $\left|A+\frac{4}{3}\right|=\frac{2}{3}$, then $z=z_{2,3}$ are parabolic fixed points*
3. *If $A\neq 0$ and $\left|A+\frac{4}{3}\right|>\frac{2}{3}$, then $z=z_{2,3}$ are repulsive fixed points.*

Proof. From (3.7),

$$|S'(z_{2,3})| = 2\left|\frac{A+1}{A}\right| \leq 1 \Rightarrow 2|A+1| \leq |A|, \quad (A\neq 0)$$

Let $A=\alpha+i\beta$ be an arbitrary complex number. Then,

$$|A+1|^2 = (\alpha+1)^2+\beta^2$$

and

$$|A|^2 = \alpha^2+\beta^2$$

So

$$\left(\alpha+\frac{4}{3}\right)^2+\beta^2 \leq \left(\frac{2}{3}\right)^2$$

Therefore,

$$|S'(z_{2,3})| \leq 1 \Rightarrow \left|A+\frac{4}{3}\right| \leq \frac{2}{3}$$

Finally, if $A\neq 0$ and $\left|A+\frac{4}{3}\right|>\frac{2}{3}$, then $|S'(z_{2,3})|\geq 1$ and $z=z_{2,3}$ are repulsive fixed points. □

Figure 1 graphically depicts the stability functions given by

$$\begin{aligned} S_1(1) &= min\left\{|S'(1)|, 1\right\} \\ S_1(z_{2,3}) &= min\left\{|S'(z_{2,3})|, 1\right\} \end{aligned}$$

The regions of stability are when $S_1(A) = 1$.

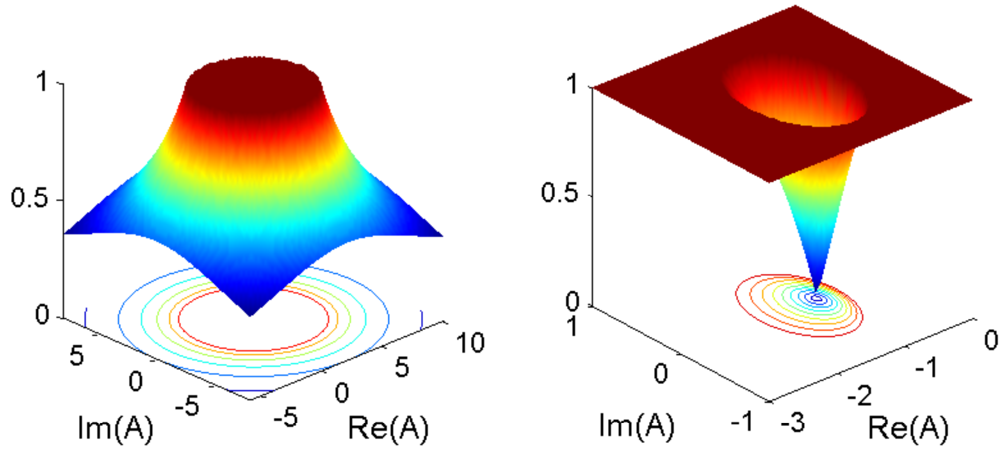

Figure 1: Stability regions of the strange fixed points. Left: $S_1(1)$. Right: $S_1(z_{2,3})$

3.3 Study of the critical points

Critical points of $S(z)$ satisfy $S'(z) = 0$, that is, $z = 0$, $z = -1$ (with multiplicity two), $z = \infty$ and

$$zc_1 = \frac{A+1+\sqrt{3A(2-A)}}{2A-1} \tag{3.8}$$

$$zc_2 = \frac{A+1-\sqrt{3A(2-A)}}{2A-1} \tag{3.9}$$

Observe that $zc_1 = \frac{1}{zc_2}$. Also, $zc_1 = zc_2 = 1$ only when $A = 2$ and $zc_1 = zc_2 = -1$ only when $A = 0$.
In Figure 2, the behavior of the fixed points and critical points for real values of A between -3 and 3 are shown. Fixed points z_2 and z_3 are represented by a blue solid line and red solid line respectively. Critical points zc_1 and zc_2 are represented by yellow dashed and green dashed lines respectively.

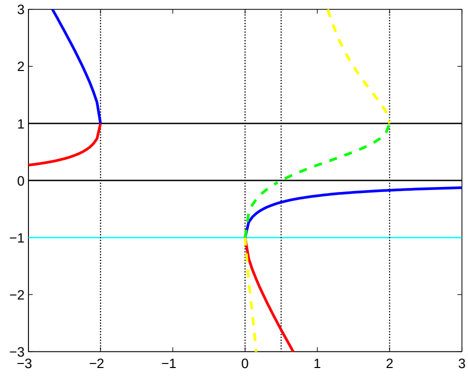

Figure 2: Dynamical Behavior of strange fixed points and critical points for $-3 < A < 3$

3.4 Study of parameter space

In this section, the behavior of the iterative methods obtained for various values of parameter A when it is used in the calculation of the critical points that are used as initial iteration is analyzed graphically. In this way, some members of the family of methods presented with good or bad behavior can be identified. In this study, we use a mesh of 1000×1000 points, a tolerance of 10^{-2} and a maximum of 50 iterations.
If the iteration begins with the critical point obtained by substituting the value of parameter A in the method for that parameter value and observing the convergence to $z = 0$ or to $z = \infty$ with the established tolerance, point A of the complex plane is represented in Figure 3 in red color. When the critical point generates iterates that do not converge, the point A is represented in blue; other colors indicate convergence to strange fixed points. The various tonalities are related to the speed of convergence; so, if the color is darker the method for that parameter value converges faster.

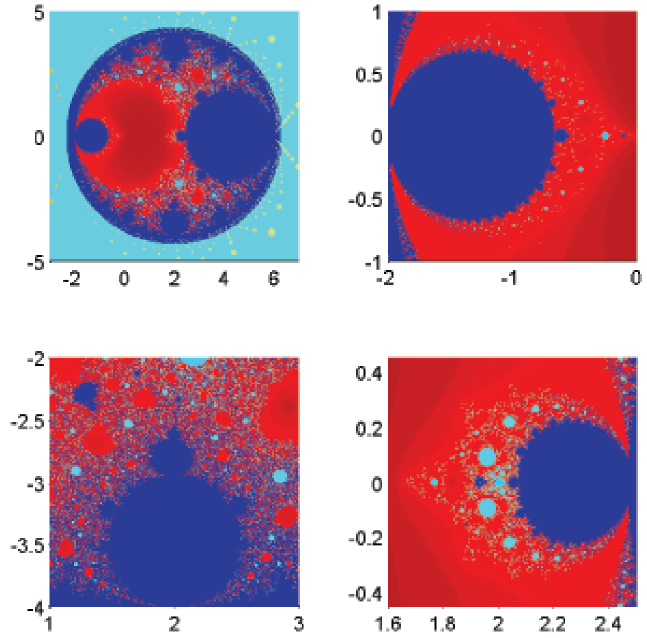

Figure 3: Parameter plane associated to the critical point $zc1$ and diverse zooms.

3.5 Dynamical Planes

In this section, the dynamic planes are represented for various methods obtained by substituting some values of parameter A in the rational function S given in (3.4). These values of A were selected from different areas of the parameter space studied in the previous section. In these dynamical planes, the convergence to 0 appear in light blue, in red appears the convergence to ∞, in dark blue the zones with no convergence to the roots and other colors show the convergence to strange fixed points. The various tonalities are related to the speed of convergence; so, if the color is darker the method converges more slowly.

In Figure 4 diverse stable dynamical planes are shown. In Figure 5 dynamical plane for $A = 2$ is shown. Other cases

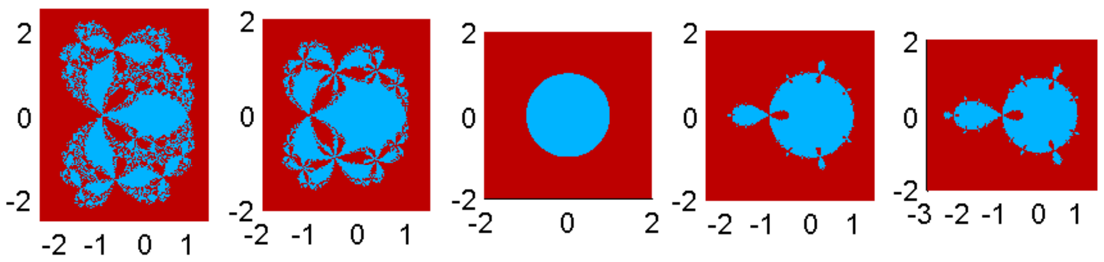

Figure 4: Dynamical planes. $A = -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}$.

of interesting dynamic planes are presented in the figures 6 and 7.

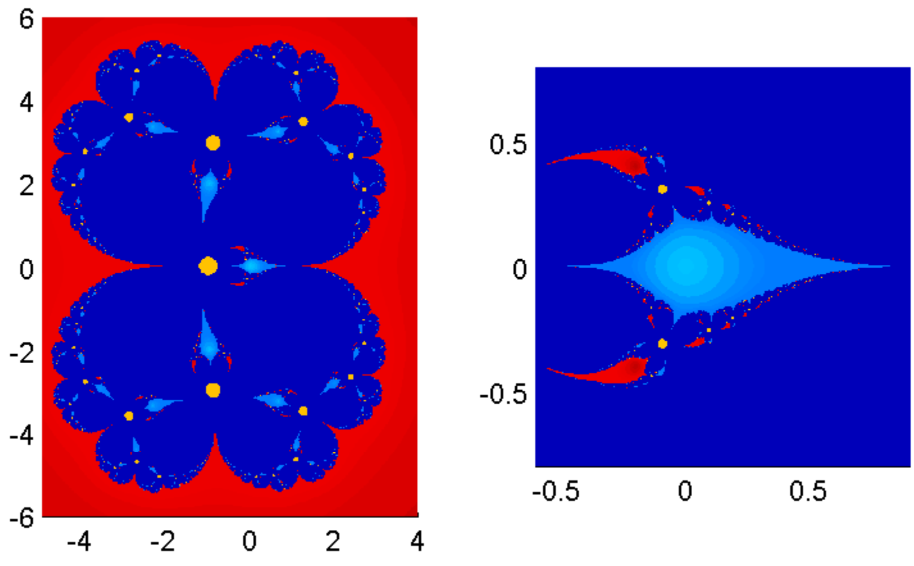

Figure 5: Dynamical planes. Left: $A = -2$. Right: zoom

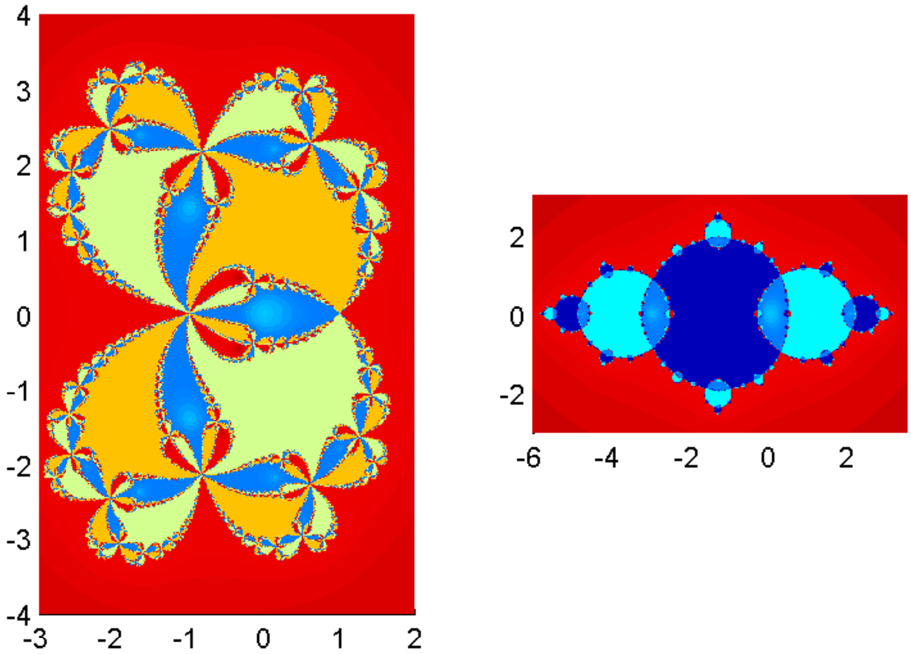

Figure 6: Dynamical planes. Left: $A = -1$. Right: $A = 2$.

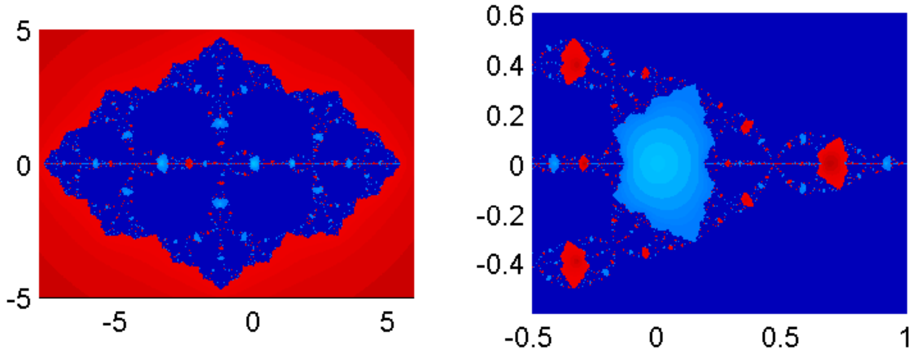

Figure 7: Dynamical planes. Left: $A = 3$. Right: zoom

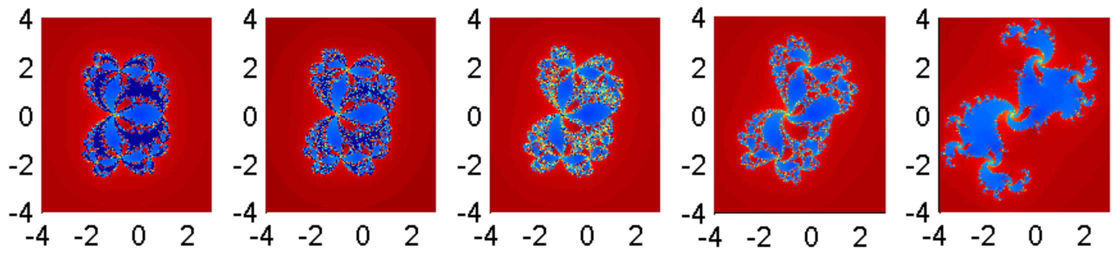

Figure 8: Dynamical planes. $A = -\frac{2}{3}, -\frac{2}{3} + i\frac{1}{10}, -\frac{2}{3} + i\frac{1}{5}, -\frac{2}{3} + i\frac{1}{2}, -\frac{2}{3} + i.$

4 Final remarks

In this paper, we present a family of Newton-Halley type methods and then a study of the complex dynamics for this family for the second-degree polynomial class is made. For this, the scaling theorem and the conjugation mapping for that family were first established, then the fixed points and critical points of the obtained rational operator were studied. We also analyzed the parameter space, selecting different values of this parameter to make the respective dynamic planes. Thus dynamic planes of methods with stable, unstable behavior and with convergence to strange fixed points are presented. It is clear that more studies on the dynamics of this family are necessary.

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