

# A family of Newton-Halley type methods to find simple roots of nonlinear equations and their dynamics

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## Abstract

In this work a new family of Newton-Halley type methods for solving nonlinear equations is presented. the dynamics of the Newton-Halley family is analyzed for the class of quadratic polynomials and the convergence is established. We find the fixed and critical points. The stable and unstable behaviors are studied. The parameter space associated with the family is studied and finally, some dynamical planes that show different aspects of the dynamics of this family are presented.

**Keywords:** Newton's method, Halley's method, order of convergence, nonlinear equations, dynamic, quadratic polynomials, Newton-Halley family.

## 1 Introduction

Iterative methods are usually necessary for solving scalar nonlinear equations. Several good methods exist in the literature: Newton, Halley and Chebyshev methods among others, see ([1]-[3]). The study of the dynamics of various methods was also done, see [4] for example.

In this paper, we give a new family of Newton-Halley type methods for solving scalar nonlinear equations. Here the author establishes the conjugacy class and when this family is applied to the class of quadratic polynomials, fixed and critical points of this family are obtained. Dynamical planes for different values of the parameter  $A$  selected from the parameter space are presented. To conclude this section, some preliminary basics are presented. Then, in section 2 the mentioned family and their convergence is presented. Subsequently, in section 3, results on the dynamics of the Newton-Halley family, with an emphasis on the stability of fixed points for then use the parameter space and thus represent the dynamic planes for different values of the parameter  $A$ . Finally, the concluding remarks are presented.

### 1.1 Basic preliminaries

We now recall some preliminaries of complex dynamics (see [4], [13] and [51]) that we use in this work. Given a rational function  $R : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ , where  $\widehat{\mathbb{C}}$  is the Riemann sphere

**Definition 1.1.** For  $z \in \widehat{\mathbb{C}}$  we define its orbit as the set  $orb(z) = \{z, R(z), R^2(z), \dots, R^n(z), \dots\}$ .

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**Definition 1.2.** A periodic point  $z_0$  of period  $m > 1$  is a point such that  $R^m(z_0) = z_0$  y  $R^k(z_0) \neq z_0$ , for  $k < m$ .

**Definition 1.3.** A pre-periodic point is a point  $z_0$  that is not periodic but exist a  $k > 0$  such that  $R^k$  is periodic.

**Definition 1.4.** A point  $z_0$  is a fixed point of  $R$  if  $R(z_0) = z_0$ .

**Definition 1.5.** A critical point  $z_{cr}$  is a point such that  $R'(z_{cr}) = 0$ .

**Definition 1.6.** A fixed point  $z_0$  is called attractor if  $|R'(z_0)| < 1$ , repulsive if  $|R'(z_0)| > 1$ , and parabolic or neutral if  $|R'(z_0)| = 1$ . If  $|R'(z_0)| = 0$  then the fixed point is called superattractor. A superattractor fixed point is also a critical point.

**Definition 1.7.** A fixed point  $z_0$  that is not associated to the roots of the function  $f(z)$  is called strange fixed point.

**Definition 1.8.** The basin of attraction of a attractor  $\alpha \in \hat{\mathbb{C}}$  is defined as the set of starting points whose orbits tend to  $\alpha$ .

## 2 Newton-Halley type methods and their Convergence

In this section we present the new family and its convergence. We recall that a sequence  $\{x_n\}_{n \geq 0}$  converges to  $r$  with order of convergence  $p$  if there exists a  $K(r) > 0$  such that

$$\lim_{n \rightarrow \infty} \frac{|x_{n+1} - r|}{|x_n - r|^p} = K(r)$$

and the error equation is

$$e_{n+1} = K(r)e_n + O(e_n^{n+1})$$

where  $e_n = x_n - r$  and  $K(r)$  is the asymptotic constant error (see).

In 1993 Hernández and Salanova [52] developed a family of Chebyshev-Halley type methods. Here, we present an uni-parametric family that allows us to study the evolution of the dynamics of the Newton-Halley family given by

$$z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)} \frac{1}{1 - AL_f(z_n)}; \quad n = 0, 1, 2, \dots \quad \text{where} \quad L_f(z) = \frac{f(z)f''(z)}{(f'(z))^2} \quad (2.1)$$

and where parameter  $A$  is complex and  $L_f$  is degree of logarithmic convexity of  $f$  (see [53]-[55]). This family includes Newton's method for  $A = 0$  and Halley's method for  $A = \frac{1}{2}$ .

To begin the study of this family, we present the convergence in the following theorem.

**Theorem 2.1.** Let  $\alpha \in I$  be a simple root of a sufficiently differentiable function  $f : B \rightarrow \mathfrak{R}$  for an open interval  $B$ . If  $x_0$  is sufficiently close to  $\alpha$ , then the family Newton-Halley type methods defined by (2.1) has almost second-order convergence, and satisfies the error equation:

$$e_{n+1} = (1 - 2A)B_2e_n^2 + 2((1 - 3A)B_3 - (2A^2 - 4A + 1)B_2^2)e_n^3 + O(e_n^4) \quad (2.2)$$

where  $e_n = z_n - \alpha$  is the error in the  $n$ th iterate and  $B_j = \frac{f^{(j)}(\alpha)}{j!f'(\alpha)}$ ,  $j = 1, 2, \dots$ .

*Proof.* By Taylor series expansion around the simple root  $\alpha$  in the  $n$ th iteration, we have

$$\begin{aligned} f(z_n) &= e_n + B_2e_n^2 + B_3e_n^3 + O(e_n^4) \\ f'(z_n) &= 1 + 2B_2e_n + 3B_3e_n^2 + 4B_4e_n^3 + O(e_n^4) \\ f''(z_n) &= 2B_2e_n + 6B_3e_n + 12B_4e_n^2 + 20B_5e_n^3 + O(e_n^4) \end{aligned}$$

Furthermore, it can be easily found by substituting these terms in (2.1) that

$$\begin{aligned} \frac{f(z_n)}{f'(z_n)} &= e_n - B_2e_n^2 + (2B_2^2 - 2B_3)e_n^3 + O(e_n^4) \\ L_f(z_n) &= 2B_2e_n + 6(B_3 - B_2^2)e_n^2 + 4(4B_2^3 - 7B_2B_3 + 3B_4)e_n^3 + O(e_n^4) \end{aligned}$$

which gives (2.2). This proves the theorem □

So, the family has order of convergence two, except for Halley's method which has order three.

### 3 Dynamical behavior of the rational function associated with Newton-Halley family

Here the author establishes the conjugacy class and the analytical expressions for the fixed and critical points of the Newton-Halley family in terms of the parameter  $A$ . Then the study of the fixed points, critical points and parameter space are presented. To finish this section several dynamical planes for different values of  $A$  selected from the parameter space are shown.

#### 3.1 Conjugacy classes

In what remains of this paper we study the dynamics of the rational map  $R$  arising from Newton-Halley family (2.1)

$$R_f = z - \frac{f(z)}{f'(z)} \frac{1}{1 - AL_f(z)}; \quad \text{where} \quad L_f(z) = \frac{f(z)f''(z)}{(f'(z))^2} \quad (3.3)$$

applied to  $P_2(z) = a_2(z - z_1)(z - z_2)$ . Let us first remember the following definition.

**Definition 3.1.** [56]. Let  $f$  and  $g$  be two maps from the Riemann sphere into itself. An analytic conjugacy between  $f$  and  $g$  is an analytic diffeomorphism  $h$  from the Riemann sphere onto itself such that  $h \circ f = g \circ h$ .

$R_f$  has the following property for an analytic function  $f$

**Theorem 3.1.** (The Scaling Theorem). Let  $f(z)$  be an analytical function on the Riemann sphere, and let  $T(z) = \alpha z + \beta$ ,  $\alpha \neq 0$ , be an affine map. If  $g(z) = f \circ T(z)$ , then  $T \circ R_g \circ T^{-1} = R_f(z)$ . That is,  $R_f$  is analytically conjugate to  $R_g$  by  $T$ .

*Proof.* With the iteration function  $R(z)$ , we have

$$R_g(T^{-1}(z)) = T^{-1}(z) - \frac{g(T^{-1}(z))}{g'(T^{-1}(z))} \frac{1}{1 - AL_g(T^{-1}(z))} \quad \text{with} \quad L_g(T^{-1}(z)) = \frac{g(T^{-1}(z))g''(T^{-1}(z))}{(g'(T^{-1}(z)))^2}$$

Since  $\alpha T^{-1}(z) + \beta = z$ ,  $g \circ T^{-1}(z) = f(z)$  and  $(g \circ T^{-1})'(z) = \frac{1}{\alpha} g'(T^{-1}(z))$ , we get  $g'(T^{-1}(z)) = \alpha (g \circ T^{-1})'(z) = \alpha f'(z)$ ,  $g''(T^{-1}(z)) = \alpha^2 f''(z)$ . We therefore have

$$\begin{aligned} T \circ R_g \circ T^{-1}(z) &= T(R_g(T^{-1}(z))) = \alpha R_g(T^{-1}(z)) + \beta \\ &= \alpha T^{-1}(z) - \frac{\alpha g(T^{-1}(z))}{g'(T^{-1}(z))} \frac{1}{1 - A \frac{g(T^{-1}(z))g''(T^{-1}(z))}{(g'(T^{-1}(z)))^2}} + \beta = z - \frac{f(z)}{f'(z)} \frac{1}{1 - A \frac{f(z)f''(z)}{(f'(z))^2}} = R_f(z) \end{aligned}$$

□

Theorem 3.1 allows the study of the dynamics of the iteration function of Newton-Halley family (2.1) for the polynomial  $P_2(z) = a_2(z - z_1)(z - z_2)$  by means of the study of the polynomial  $p(z) = (z - a)(z - b)$  where  $a \neq b$ .

**Definition 3.2.** [10]. We say that a one-point iterative root-finding algorithm  $p \rightarrow T_p$  has a universal Julia set (for polynomials of degree  $d$ ) if there exists a rational map  $S$  such that for every degree  $d$  polynomial  $p$ ,  $J(T_p)$  is conjugate by a Möbius transformation to  $J(S)$

The following theorem establishes a universal Julia set for quadratics for our method (2.1).

**Theorem 3.2.** For a rational map  $R_p(z)$  given by (2.1) applied to  $p(z) = (z - a)(z - b)$ ,  $a \neq b$ ,  $R_p(z)$  is conjugate via the Möbius transformation given by  $M(z) = \frac{z-a}{z-b}$  to

$$S(z) = \frac{z^2(z + 1 - 2A)}{(1 - 2A)z + 1} \quad (3.4)$$

*Proof.* Let  $p(z) = (z-a)(z-b)$ ,  $a \neq b$  and let  $M(z) = \frac{z-a}{z-b}$  with  $M^{-1}(u) = \frac{bu-a}{u-1}$ . We then have

$$M \circ R_p \circ M^{-1}(z) = M \left( R_p \left( \frac{bz-a}{z-1} \right) \right) = \frac{z^2(z+1-2A)}{(1-2A)z+1}$$

□

We observe that parameters  $a$  and  $b$  do not appear in  $S(z)$ , because the Newton-Halley family complies with theorem 3.2.

The next subsections consider three specific values of  $A$

### 3.1.1 $A = 0$ : *Newton's Method*

In this case  $S(z) = z^2$  and the fixed points are  $z = 0$ ,  $z = 1$  and  $z = \infty$ . As  $S'(z) = 2z$  then  $|S'(0)| = 0$ ,  $|S'(1)| = 2$  and  $|S'(\infty)| = \infty$ . So,  $z = 0$  and  $z = \infty$  are superattractive fixed points.  $z = 1$  is a repulsive strange fixed point.

### 3.1.2 $A = \frac{1}{2}$ : *Halley's Method*

If  $A = \frac{1}{2}$  then  $S(z) = z^3$  and the fixed points are  $z = 0$ ,  $z = \pm 1$  and  $z = \infty$ . As  $S'(z) = 3z^2$  then  $|S'(0)| = 0$ ,  $|S'(\pm 1)| = 3$  and  $|S'(\infty)| = \infty$ . So,  $z = 0$  and  $z = \infty$  are superattractive fixed points.  $z = \pm 1$  are repulsive strange fixed points.

### 3.1.3 $A = 1$ : *Newton's method for multiple roots*

If  $A = 1$  then  $S = -z^2$  with fixed points  $z = 0$ ,  $z = -1$  and  $z = \infty$ . As  $S'(z) = -2z$  then  $|S'(0)| = 0$ ,  $|S'(-1)| = 2$  and  $|S'(\infty)| = \infty$ . So,  $z = 0$  and  $z = \infty$  are superattractive fixed points.  $z = -1$  is a repulsive strange fixed point.

## 3.2 Study of the fixed points

The fixed points of  $S$  for  $S$  defined in (3.4) are  $z = -1$ ,  $z = 0$ ,  $z = 1$  and  $z = \infty$ .

To study the stability of the fixed points, we calculate  $S'(z)$ , so

$$S'(z) = \frac{2z[(1-2A)z^2 + 2(A^2 - A + 1)z + 1 - 2A]}{[(1-2A)z + 1]^2} \quad (3.5)$$

It is obvious from (3.5) that  $z = 0$  and  $z = \infty$  are superattractive fixed points. The study of stability of the other fixed points is now presented.

The operator  $S'(z)$  in  $z = -1$  gives

$$|S'(-1)| = \left| \frac{A+1}{A} \right|, \quad (A \neq 0) \quad (3.6)$$

If we analyze this function, we obtain an horizontal asymptote in  $|S'(-1)| = 1$  when  $A \rightarrow \pm\infty$ , and a vertical asymptote in  $A = 0$  (Newton's method).

In the following result we present the stability of the fixed point  $z = -1$ .

**Theorem 3.3.** *The strange fixed point  $z = -1$  satisfies the following statements:*

1. *If  $\text{Re}\{A\} < -\frac{1}{2}$ , then  $z = -1$  is an attractor and is a superattractor for  $A = -1$ .*
2. *If  $\text{Re}\{A\} = -\frac{1}{2}$ , then  $z = -1$  is a parabolic fixed point.*
3. *If  $A \neq 0$  and  $\text{Re}\{A\} > -\frac{1}{2}$ , then  $z = -1$  is a repulsive fixed point.*

*Proof.* From (3.6),

$$|S'(-1)| = \left| \frac{A+1}{A} \right| \leq 1 \Rightarrow |A+1| \leq |A|$$

Let  $A = \alpha + i\beta$  be an arbitrary complex number. Then,

$$|A+1|^2 = (\alpha+1)^2 + \beta^2$$

and

$$|A|^2 = \alpha^2 + \beta^2$$

So

$$2\alpha + 1 \leq 0 \Rightarrow \alpha \leq -\frac{1}{2}$$

Therefore,

$$|S'(-1)| \leq 1 \Leftrightarrow \operatorname{Re}\{A\} \leq -\frac{1}{2}$$

Finally, if  $A \neq 0$  and  $\operatorname{Re}\{A\} > -\frac{1}{2}$ , then  $|S'(-1)| \geq 1$ . □

The operator  $S'(z)$  in  $z = 1$  gives

$$|S'(1)| = \left| \frac{A-2}{A-1} \right|, \quad (A \neq 1) \quad (3.7)$$

If we analyze this function, we obtain an horizontal asymptote in  $|S'(1)| = 1$  when  $A \rightarrow \pm\infty$ , and a vertical asymptote in  $A = 1$  (Newton's method for multiple roots).

In the following result we present the stability of the fixed point  $z = 1$ .

**Theorem 3.4.** *The strange fixed point  $z = 1$  satisfies the following statements:*

1. *If  $\operatorname{Re}\{A\} > \frac{3}{2}$ , then  $z = 1$  is an attractor and it is a superattractor for  $A = 2$ .*
2. *If  $\operatorname{Re}\{A\} = \frac{3}{2}$ , then  $z = 1$  is a parabolic fixed point.*
3. *If  $A \neq 1$  and  $\operatorname{Re}\{A\} < \frac{3}{2}$ , then  $z = 1$  is a repulsive fixed point.*

*Proof.* From (3.7),

$$|S'(1)| = \left| \frac{A-2}{A-1} \right| \leq 1 \Rightarrow |A-2| \leq |A-1|$$

So

$$2\alpha - 3 \geq 0 \Rightarrow \alpha \geq \frac{3}{2}$$

Therefore,

$$|S'(1)| \geq 1 \Leftrightarrow \operatorname{Re}\{A\} \geq \frac{3}{2}$$

Finally, if  $A \neq 1$  and  $\operatorname{Re}\{A\} < \frac{3}{2}$ , then  $|S'(1)| \geq 1$ . □

In Figure 1 the functions where are observed the regions of stability are graphed. These functions are given by

$$\begin{aligned} S_1(-1) &= \min\{|S'(-1)|, 1\} \\ S_1(1) &= \min\{|S'(1)|, 1\} \end{aligned}$$

Zones of stability are when  $S_1(A) = 1$ .

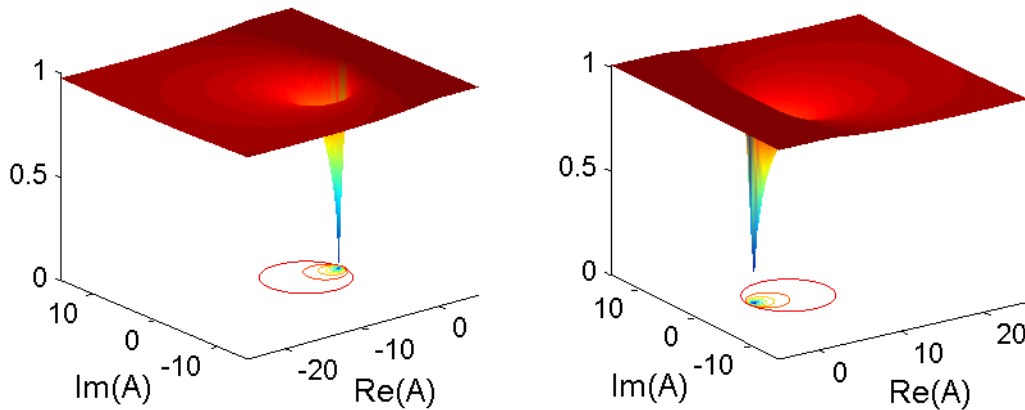


Figure 1: Stability regions of the strange fixed points. Left:  $S_1(-1)$ . Right:  $S_1(1)$

3.3 Study of the critical points

Critical points of  $S(z)$  satisfy  $S'(z) = 0$ , that is,  $z = 0, z = \infty$  and

$$zc_1 = \frac{A^2 - A + 1 + \sqrt{A^4 - 2A^3 - A^2 + 2A}}{2A - 1} \tag{3.8}$$

$$zc_2 = \frac{A^2 - A + 1 - \sqrt{A^4 - 2A^3 - A^2 + 2A}}{2A - 1} \tag{3.9}$$

if  $A \neq 0, \frac{1}{2}, 1$ . Observe that  $zc_2 = \frac{1}{zc_1}$  and  $zc_1 = zc_2 = 1$  only when  $A = 2$ .  $zc_1 = zc_2 = -1$  only when  $A = -1$ . When  $A = 0$  or  $A = 1$  the only one critical point is  $z = 0$  and if  $A = \frac{1}{2}$ ,  $z = 0$  is a critical point with multiplicity two. In Figure 2, the author represent the behavior of the fixed points and critical points for real values of  $A$  between  $-4$  and  $4$ . Fixed points are represented by black solid lines and this is more thick when fixed points are attractors. Critical points  $zc_1$  and  $zc_2$  are represented by red solid line and blue dotted line respectively.

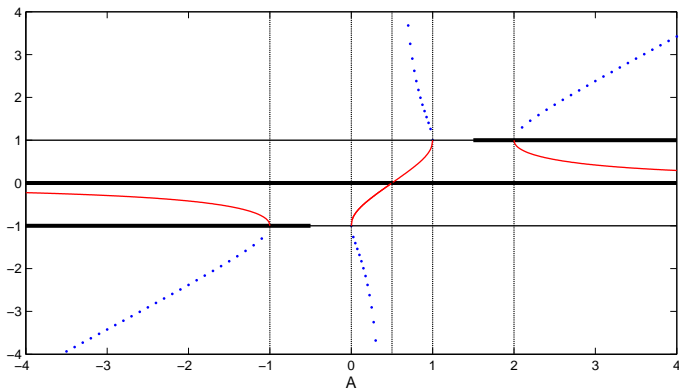


Figure 2: Dynamical Behavior of strange fixed points and critical points for  $-4 < A < 4$

3.4 Study of parameter space

In this section the behavior of the iterative methods obtained for various values of parameter  $A$  when it is used in the calculation of the critical points that are used as initial iteration is analyzed graphically. In this way some

members of the family of methods presented with good or bad behavior can be identified. In this study, we use a mesh of  $1000 \times 1000$  points, a tolerance of  $10^{-2}$  and a maximum of 50 iterations.

If the iteration begins with the critical point obtained by substituting the value of parameter A in the method for that parameter value and observing the convergence to  $z = 0$  or to  $z = \infty$  with the established tolerance, point A of the complex plane is represented in Figure 3 in red color. When the critical point generates iterates that do not converge, the point A is represented in blue; other colors indicate convergence to strange fixed points. The various tonalities are related to the speed of convergence; so, if the color is darker the method for that parameter value converges faster. Figure 3 on the right shows a zoom to observe in more detail the behavior of the method in non-convergence zones.

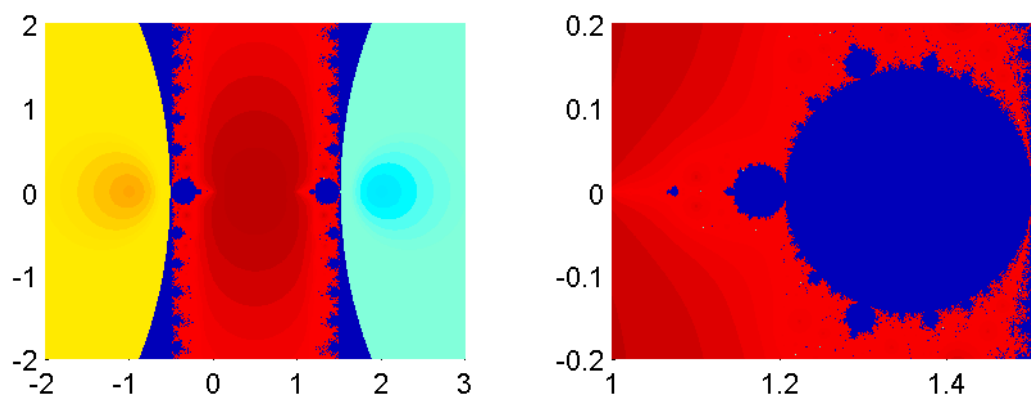


Figure 3: Parameter plane associated to the critical point  $zc1$  and zoom

3.5 Dynamical Planes

In this section the dynamic planes are represented for various methods obtained by substituting some values of parameter A in the rational function S given in (3.4). These values of A were selected from different areas of the parameter space studied in the previous section. In these dynamical planes the convergence to 0 appear in light blue, in red appears the convergence to  $\infty$ , in dark blue the zones with no convergence to the roots and other colors show the convergence to strange fixed points. The various tonalities are related to the speed of convergence; so, if the color is darker the method converges more slowly.

Now, in Figures 4-7 various stable dynamic planes for values of A selected in the parameter space are showed.

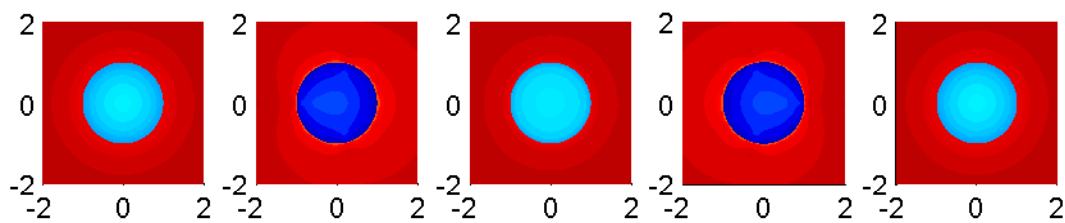


Figure 4: Dynamical planes. Left: Newton’s method ( $A = 0$ ). Second Left:  $A = \frac{1}{4}$ . Center: Halley’s method ( $A = \frac{1}{2}$ ). Second right:  $A = \frac{3}{4}$ . Right:  $A = 1$

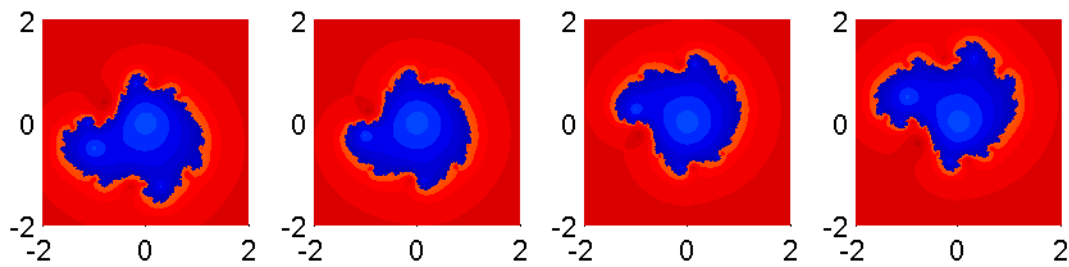


Figure 5: Dynamical planes. Left: Newton's method ( $A = -\frac{i}{4}$ ). Second Left:  $A = -\frac{i}{8}$ . Right:  $A = \frac{i}{8}$ . Right:  $A = \frac{i}{4}$

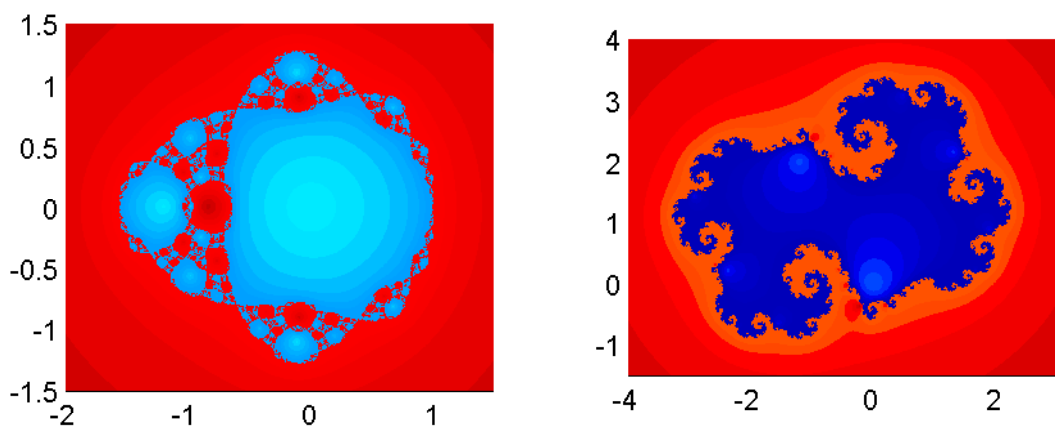


Figure 6: Dynamical planes. Left:  $A = -0.1$ . Right:  $A = -0.1 + i$

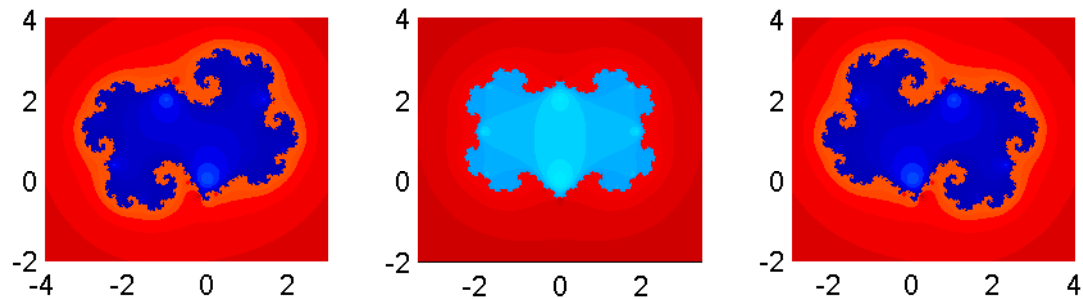


Figure 7: Dynamical planes with complex parameters. Left:  $A = i - \frac{1}{2}$ . Center:  $A = i$ . Right:  $A = i + 1$ .

In Figures 8-9, the dynamical planes of several members of the family with broad regions of no convergence is shown.



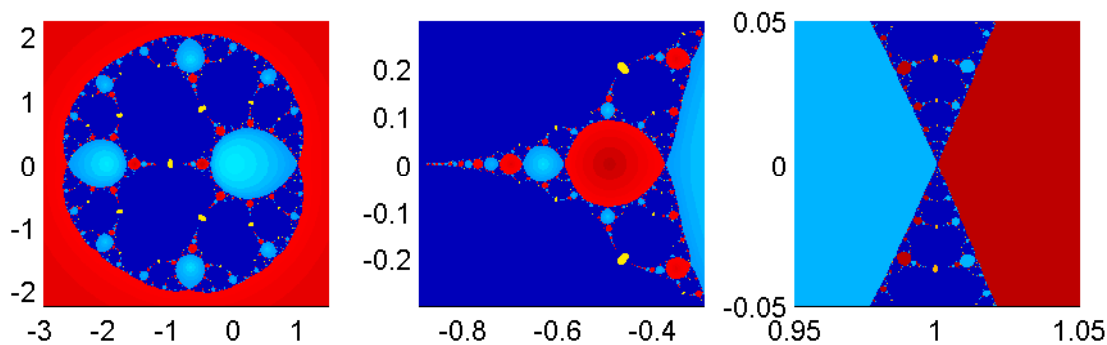


Figure 8: Dynamical planes. Left:  $A = -\frac{1}{2}$ . Center: zoom in  $(-0.9,-0.3)$ . Right: zoom in  $(0.95,1.05)$

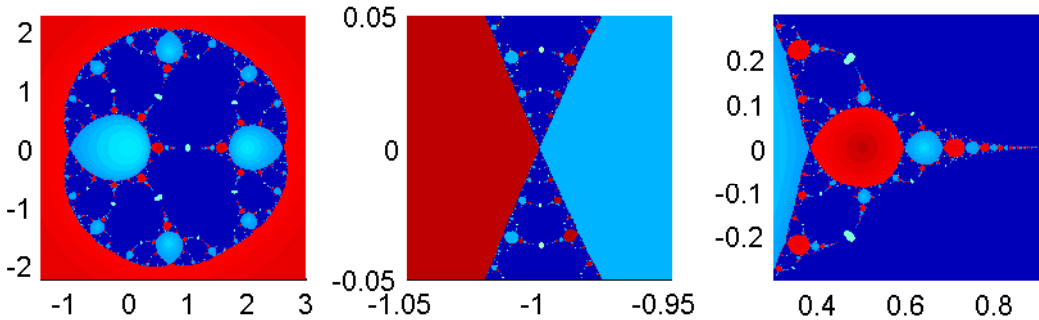


Figure 9: Dynamical planes. Left:  $A = \frac{3}{2}$ . Center: zoom in  $(-1.05,-0.95)$ . Right: zoom in  $(0.3,0.9)$

In Figures 10-11, the dynamical planes of several members of the family with regions of convergence to any of the strange fixed points is shown.

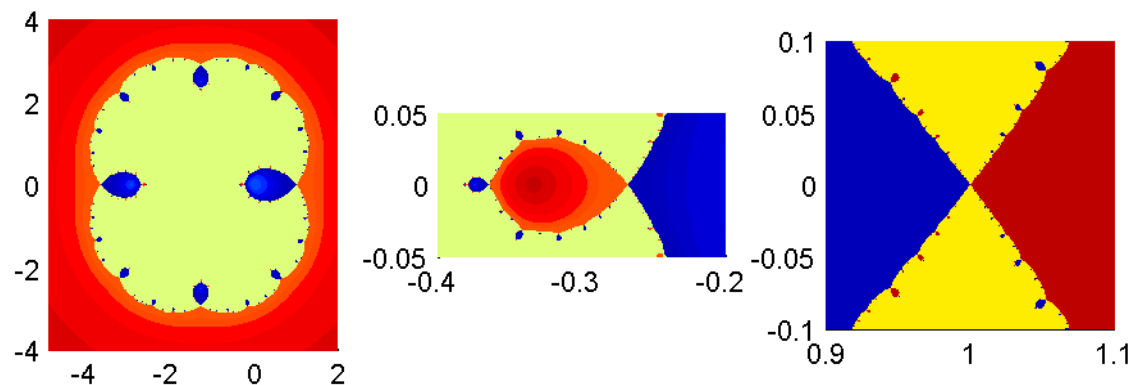


Figure 10: Dynamical planes. Left:  $A = -1$ . Center: zoom in  $(-0.4,-0.2)$ . Right: zoom in  $(0.9,1.1)$

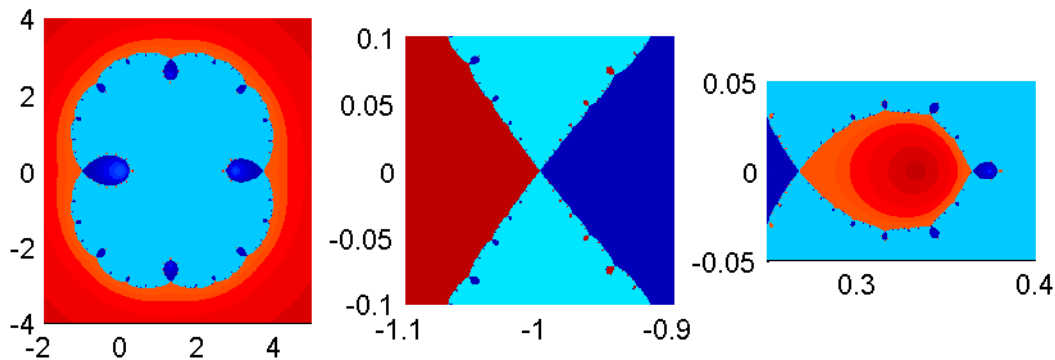


Figure 11: Dynamical planes. Left:  $A = 2$ . Center: zoom in  $(-1.1,-0.9)$ . Right: zoom in  $(0.25,0.4)$

3.6 Periodic orbits

In this section we present different periodic orbits of period two and only one periodic orbit of period three for  $A = 1.35$ . First,  $S(S(z)) = z$  is resolved, where  $S$  is give in (3.4). In Figure 12 left, dynamical plane to  $A = 1.35$  is presented, where the three orbits of period two can be observed. Such orbits are given by

$$z_1 = 0.37830094 + i0.37830094 \Rightarrow z_2 = 0.37830094 - i0.37830094 \Rightarrow z_1 = 0.37830094 + i0.37830094$$

$$z_3 = 0.85 + i0.5267826876 \Rightarrow z_4 = 0.85 - i0.5267826876 \Rightarrow z_3 = 0.85 + i0.5267826876$$

$$z_5 = 1.32169906 + i1.32169906 \Rightarrow z_6 = 1.32169906 - i1.32169906 \Rightarrow z_5 = 1.32169906 + i1.32169906$$

To calculate periodic orbits of period three is necessary that  $S(S(S(z))) = z$ . In this case, eight orbits can be obtained. In Figure 12 right, dynamical plane to  $A = 1.35$  is presented, where a orbit of period three can be observed. Such orbit is given by

$$z_1 = 0.75555 + i0.080826 \Rightarrow z_2 = 1.6435 - i0.5488 \Rightarrow z_3 = 0.216577 + i0.79 \Rightarrow z_1 = 0.75555 + i0.080826$$

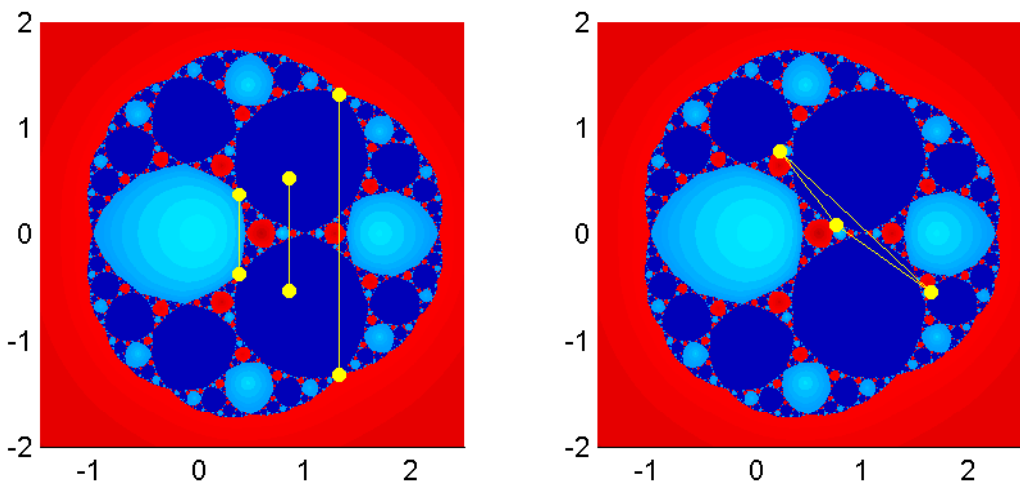


Figure 12: Dynamical planes for  $A = 1.35$ . Left: 2-periodic orbit. Right: 3-periodic orbit.

## 4 Result and discussion

In this paper we present a family of Newton-Halley type methods and then a study of the complex dynamics for this family for the second-degree polynomial class is made. For this, the scaling theorem and the conjugation mapping for that family were first established, then the fixed points and critical points of the obtained rational operator were studied. We also analyzed the parameter space, selecting different values of this parameter to make the respective dynamic planes. Thus dynamic planes of methods with stable, unstable behavior and with convergence to strange fixed points are presented. Finally, we show the existence of periodic orbits, representing graphically all orbits of period two and one of the eight orbits of period three. It is clear that more studies on the dynamics of this family are necessary.

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