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# On Several Gander's Theorem Based Third-Order Iterative Methods for Solving Nonlinear Equations and Their Geometric Constructions

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**Abstract**. *In this paper new families of methods based on Gander's theorem for solving nonlinear equations of third-order are presented. The asymptotic error constant is derived and a geometric construction is provided for a theorem to define these families. Some numerical tests are presented in order to demonstrate the good performance of some members of these families.*

**Key words** : Nonlinear Equations, Iterative Methods, Third-Order, Asymptotic Error Constant, Geometric Construction.

**AMS Subject Classifications** : 65H05

## 1. Introduction

Is well known that nonlinear equations are in general unsolvable analytically and the solution to these equations must be approached using iterative methods. Newton's method is the iterative method most used to solve nonlinear equations. It has order of convergence two and is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad n = 0,1,2,\cdots \tag{1}$$

Construction of these iterative methods when they are of third-order is by using a Gander's result [10]. Three of the most popular methods for a point with third-order convergence are Chebyshev ([20],[21]), Halley [13] and Super-Halley's methods ([12],[15]). Also, various families of methods using weight functions that meet the hypothesis of Gander's theorem can be constructed. These may include Hansen-Patrick [14], Chebyshev-Halley [11] and $\theta - C$ [2] families of iterative methods.

Several authors, ([1],[16],[18],[23]), have presented different constructions and geometric interpretations of third-order methods. In this work, we establish a geometric interpretation of methods that can be based on Gander's theorem.

The rest of the paper is organized as follows. In section 2 Schröder's theorem is presented to be used to demonstrate the extended Gander's theorem. Then, in section 3 the main result of this work is presented. In section 4 several examples of families with their geometric interpretation and asymptotic error constant are reported. Finally, in section 5, numerical comparisons are made to show the performance of some of the methods presented.

## 2. Gander's Theory

### 2.1. **Schröder's theorem**

In this section we first present Schröder's theorem that will be useful in the sequel. We recall that a sequence $\{x_n\}_{n\geq 0}$ converges to $\alpha$ with order of convergence $r$ if there exists a $K(\alpha) > 0$ such that

$$\lim_{n\to\infty} \frac{|x_{n+1}-\alpha|}{|x_{n+1}-\alpha|^r} = K(\alpha),$$

Moreover, if $e_n = x_n - \alpha$ is the error in the $n-$ th iteration and $K(\alpha)$ is the asymptotic error constant then the relation

$$e_{n+1} = K(\alpha)e_n^r + O(e_n^{r+1})$$

is the error equation.

**Theorem** [24] **2.1**. (Schröder's theorem ) *Let the iterative method be given by the equation of iteration $x_{n+1} = G(x_n)$, where $G \in C^r$ in a neighborhood of $\alpha$, a fixed point of $G$, $\alpha = G(\alpha)$. Then the sequence $\{x_n\}$ converge to $\alpha$ with order of convergence $r$ if:*

$$\frac{d^jG(\alpha)}{dx^j} = 0, \quad j = 1,2,\cdots,r-1, \quad \frac{d^rG(\alpha)}{dx^r} \neq 0.$$

### 2.2. **Extended Gander's theorem**

As an extension of Gander's theorem [10] we report on the following result.

**Theorem 2.2**. *Let $\alpha$ be a simple zero of $f$ and $H$ be any function with $H(0) = 1$, $\dot{H}(0) = \frac{1}{2}$ and $|\ddot{H}(0)| < \infty$. If the iteration $x_{n+1} = G(x_n)$, with $G(x) = x - \frac{f(x)}{f'(x)}H(L_f(x))$ where $L_f(x) = \frac{f(x)f''(x)}{\left[f'(x)\right]^2}$, is of third-order, then the asymptotic error constant is given by*

$$K(\alpha) = \frac{1}{2}(1-\ddot{H}(0))\left[\frac{f''(\alpha)}{f'(\alpha)}\right]^2 - \frac{1}{6}\frac{f'''(\alpha)}{f'(\alpha)}. \tag{2}$$

*Proof*. For the above $G(x)$, the hypothesis of this theorem requires that $G'(\alpha) = 1 - H(0) = 0$

and $G''(\alpha) = \dfrac{f''(\alpha)}{f'(\alpha)}(H(0) - 2\dot{H}(0)) = 0$. Now, by Schröder's theorem we have that the method given by $x_{n+1} = G(x_n)$ is of third order. Furthermore

$$G'''(\alpha) = 3(1 - \ddot{H}(0))\left[\frac{f''(\alpha)}{f'(\alpha)}\right]^2 - \frac{f'''(\alpha)}{f'(\alpha)}. \tag{3}$$

Since the asymptotic error constant is $K(\alpha) = \dfrac{G'''(\alpha)}{3!}$, then by (3) we obtain (2). Here the proof ends. ■

It should be underlined here that by using Taylor's expansion, Cordero, Jordan and Torregrosa obtained in [8] an equivalent result.

## 3. Geometric Construction of the Gander's Class

Following an idea of Amat, Busquier and Gutiérrez [1], we replace $f(x_n)$ by $f(x_n) - y$ in the equation of iteration $x_{n+1} = G(x_n)$ to arrive at the following result.

**Theorem 3.1**. *Let $\alpha$ be a simple zero of $f$ and $H$ be any function with $H(0) = 1$, $\dot{H}(0) = \frac{1}{2}$ and $|\ddot{H}(0)| < \infty$. The iteration $x_{n+1} = G(x_n)$, with $G(x) = x - \dfrac{f(x)}{f'(x)}H(L_f(x))$ where $L_f(x) = \dfrac{f(x)f''(x)}{\left[f'(x)\right]^2}$, can be built from the curve defined by equation*

$$x = x_n + \frac{y - f(x_n)}{f'(x_n)}H\left(\frac{[f(x_n) - y]f''(x_n)}{\left(f'(x_n)\right)^2}\right), \tag{4}$$

*which meets the following three conditions of tangency:*
*1. $y(x_n) = f(x_n)$,*
*2. $y'(x_n) = f'(x_n)$,*
*3. $y''(x_n) = f''(x_n)$.*

*Proof.* If $y = 0$ and $x = x_{n+1}$ are used in equation (4), then $x_{n+1} = G(x_n)$ is obtained. If in equation (4), $x$ is replaced by $x_n$, then the first condition of tangency $y(x_n) = f(x_n)$ is satisfied.

Now, let $g(y(x)) = L_f(x_n)\left(1 - \dfrac{y(x)}{f(x_n)}\right)$ and $y(x_n) = f(x_n)$, then

$$g(y(x_n)) = 0, \quad g'(y(x_n)) = -\frac{f'(x_n)}{f(x_n)}L_f(x_n), \tag{5}$$

and

$$g''(y(x_n)) = -\frac{f''(x_n)}{f(x_n)}L_f(x_n). \tag{6}$$

For the sake of simplicity, we write (4) in the following form

$$\frac{f'(x_n)}{f(x_n)} L_f(x_n)(x-x_n) = -g(y(x))H(g(y(x))). \tag{7}$$

Now differentiate (7) twice with respect to the variable $x$ to obtain

$$\frac{f'(x_n)}{f(x_n)} L_f(x_n) = -g'(y(x))H(g(y(x))) - g(y(x))\dot{H}(g(y(x)))g'(y(x)) \tag{8}$$

and

$$0 = -g''(y(x))H(g(y(x))) - 2\dot{H}(g(y(x)))[g'(y(x))]^2 - g(y(x))[g'(y(x))\dot{H}(g(y(x)))]'. \tag{9}$$

If we consider $x = x_n$ in (8) and (9), using equations (5) and (6) as well as $H(0) = 1$, $\dot{H}(0) = \frac{1}{2}$ and $|\ddot{H}(0)| < \infty$, the other two tangency conditions are obtained. ■

## 4. A Compilation of Third-Order Families

In this section we present several examples in which the weight function $H$ satisfies the hypotheses of the extended Gander's theorem, thus yielding different equations with the structure of equation (4) that allow to construct third order methods. In the first three examples, three families of well-known methods, are presented to solve non-linear equations of third order when the roots are simple. Then new families, that the author believes are not in the literature, are presented. In all presented families, $x_0$ is given and $n = 0, 1, 2, \cdots$. Also presented, are the elements of the family in which $\ddot{H}(0) = 1$. These elements have order four for quadratic equations.

### 4.1. **Hansen-Patrick's family** [14]

Let the weight function $H$ be given by

$$H(t) = \frac{a+1}{a+\sqrt{1-(a+1)t}} \quad (a \neq -1) \quad where \quad H(0) = 1,\ \dot{H}(0) = \tfrac{1}{2}\ and\ \ddot{H}(0) = \frac{a+3}{4}.$$

If we consider the curve defined by equation

$$x = x_n + \frac{y-f(x_n)}{f'(x_n)} \frac{(a+1)}{\left(a+\sqrt{1-(a+1)\frac{[f(x_n)-y]f''(x_n)}{\left(f'(x_n)\right)^2}}\right)},$$

then by theorem 3.1 the tangency conditions are satisfied. So, the equation of iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \frac{a+1}{\left(a+\sqrt{1-(a+1)L_f(x_n)}\right)} \tag{10}$$

and

$$K(\alpha) = \frac{(1-a)}{8}\left[\frac{f''(\alpha)}{f'(\alpha)}\right]^2 - \frac{1}{6}\frac{f'''(\alpha)}{f'(\alpha)}.$$

This family of third-order iterative algorithms includes, as particular cases, the following methods

- When $a = 0$, we have the Ostrowski's method [21]

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \frac{1}{\sqrt{1-L_f(x_n)}}, \tag{11}$$

with

$$K(\alpha) = \frac{1}{8}\left[\frac{f''(\alpha)}{f'(\alpha)}\right]^2 - \frac{1}{6}\frac{f'''(\alpha)}{f'(\alpha)}.$$

• When $a = 1$, corresponds to the Euler's method (in this case $\ddot{H}(0) = 1$) [25]

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\frac{2}{\left(1+\sqrt{1+2L_f(x_n)}\right)}, \tag{12}$$

with

$$K(\alpha) = -\frac{1}{6}\frac{f'''(\alpha)}{f'(\alpha)}.$$

• If we let $a \to \pm\infty$, Newton's method (1) is obtained.

4.2. **Chebyshev-Halley's family** [11]

Here the weight function $H$ is given by

$$H(t) = 1 + \frac{t}{2(1-At)} \Rightarrow H(0) = 1, \dot{H}(0) = \frac{1}{2} \text{ and } \ddot{H}(0) = A.$$

If we consider the curve defined by the equation

$$x = x_n + \frac{y-f(x_n)}{f'(x_n)}\left(1 + \frac{\frac{[f(x_n)-y]f''(x_n)}{\left(f'(x_n)\right)^2}}{2\left(1-A\frac{[f(x_n)-y]f''(x_n)}{\left(f'(x_n)\right)^2}\right)}\right),$$

then by theorem 3.1 the tangency conditions are satisfied. So, the equation of iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left(1 + \frac{L_f(x_n)}{2(1-AL_f(x_n))}\right) \tag{13}$$

and

$$K(\alpha) = \frac{(1-A)}{2}\left[\frac{f''(\alpha)}{f'(\alpha)}\right]^2 - \frac{1}{6}\frac{f'''(\alpha)}{f'(\alpha)}.$$

This family of third-order iterative algorithms includes, as particular cases, the methods that follow.

• $A = 0$, corresponds to Chebyshev's method (where $\ddot{H}(0) = 0$) ([20],[21]),

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left(1 + \frac{1}{2}L_f(x_n)\right), \tag{14}$$

and

$$K(\alpha) = \frac{1}{2}\left[\frac{f''(\alpha)}{f'(\alpha)}\right]^2 - \frac{1}{6}\frac{f'''(\alpha)}{f'(\alpha)}.$$

• $A = \frac{1}{2}$, corresponds to Halley's method (in this case $\ddot{H}(0) = 1/2$) [13],

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left(\frac{1}{1-\frac{1}{2}L_f(x_n)}\right), \tag{15}$$

and

$$K(\alpha) = \frac{1}{4}\left[\frac{f''(\alpha)}{f'(\alpha)}\right]^2 - \frac{1}{6}\frac{f'''(\alpha)}{f'(\alpha)}.$$

• $A = 1$, corresponds to Super-Halley's method (where $\ddot{H}(0) = 1$) ([12],[15])

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left(\frac{2 - L_f(x_n)}{2(1 - L_f(x_n))}\right), \tag{16}$$

and

$$K(\alpha) = -\frac{1}{6}\frac{f'''(\alpha)}{f'(\alpha)}.$$

• If we let $A \to \pm\infty$, Newton's method (1) is again obtained.

It should be noted that the dynamics of the Chebyshev-Halley family on quadratic polynomials was studied in [7].

### 4.3. $\theta - C$ **family** [2]

As before, let the weight function $H$ be given by

$$H(t) = 1 + \tfrac{1}{2}\frac{t}{(1-\theta t)} + Ct^2 \Rightarrow H(0) = 1, \dot{H}(0) = \tfrac{1}{2} \textit{ and } \ddot{H}(0) = \theta + 2C.$$

For the equation

$$x = x_n + \frac{y - f(x_n)}{f'(x_n)}\left[1 + \frac{\frac{[f(x_n) - y]f''(x_n)}{(f'(x_n))^2}}{2\left(1 - \theta\frac{[f(x_n) - y]f''(x_n)}{(f'(x_n))^2}\right)} + C\left(\frac{[f(x_n) - y]f''(x_n)}{(f'(x_n))^2}\right)^2\right],$$

by theorem 3.1 the tangency conditions are satisfied. So, the equation of iteration is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left(1 + \frac{L_f(x_n)}{2(1 - \theta L_f(x_n))} + C\,(L_f(x_n))^2\right), \tag{17}$$

and

$$K(\alpha) = \frac{(1 - \theta - 2C)}{2}\left[\frac{f''(\alpha)}{f'(\alpha)}\right]^2 - \frac{1}{6}\frac{f'''(\alpha)}{f'(\alpha)}.$$

This family of third-order iterative algorithms includes, as particular cases, the following methods.

• When $\theta = 0$, we obtain the $C$ –family (in this case $\ddot{H}(0) = 2C$) [9]

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left(1 + \frac{1}{2}L_f(x_n) + C\,(L_f(x_n))^2\right), \tag{18}$$

with

$$K(\alpha) = \frac{(1 - 2C)}{2}\left[\frac{f''(\alpha)}{f'(\alpha)}\right]^2 - \frac{1}{6}\frac{f'''(\alpha)}{f'(\alpha)}.$$

The dynamics of $C$ –family (18) on quadratic polynomials is studied in [5].

• When $C = 0$, the Chebyshev-Halley's family (13) is obtained.

• When $C = \frac{1-\theta}{2}$ (in this case $\ddot{H}(0) = 1$) the following family is obtained

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left(1 + \frac{L_f(x_n)}{2(1-\theta L_f(x_n))} + \frac{(1-\theta)}{2}\,(L_f(x_n))^2\right),$$

and

$$K(\alpha) = -\frac{1}{6}\frac{f'''(\alpha)}{f'(\alpha)}.$$

The dynamics of $\theta$-C family (17) on quadratic polynomials are reported in [4] and [6] .

## 4.4. **A convex combination of the Halley and Chebyshev methods**

Using a convex combination of (14) and (15) the following $H$ function is obtained.

$$H(t) = \frac{A}{(1-\frac{1}{2}t)} + (1-A)(1+\tfrac{1}{2}t) \Rightarrow H(0) = 1,\ \dot{H}(0) = \tfrac{1}{2}\ \textit{and}\ \ddot{H}(0) = \tfrac{A}{2}.$$

For

$$x = x_n + \frac{y-f(x_n)}{f'(x_n)}\left[\frac{A}{\left(1-\frac{1}{2}\frac{[f(x_n)-y]f''(x_n)}{\left(f'(x_n)\right)^2}\right)} + (1-A)\left(1+\frac{1}{2}\frac{[f(x_n)-y]f''(x_n)}{\left(f'(x_n)\right)^2}\right)\right],$$

then by theorem 3.1, the equation of iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left(\frac{A}{1-\frac{1}{2}L_f(x_n)} + (1-A)\left(1+\frac{1}{2}L_f(x_n)\right)\right), \tag{19}$$

and

$$K(\alpha) = \frac{(2-A)}{4}\left[\frac{f''(\alpha)}{f'(\alpha)}\right]^2 - \frac{1}{6}\frac{f'''(\alpha)}{f'(\alpha)}.$$

If $A = 0$ Chebyshev's method (14) is obtained. $A = 1$ happens to correspond to Halley's family (15). While when $A = 2$ in (19) we may invoke the following iteration equation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left(\frac{2}{1-\frac{1}{2}L_f(x_n)} - 1 - \frac{1}{2}L_f(x_n)\right), \tag{20}$$

and

$$K(\alpha) = -\frac{1}{6}\frac{f'''(\alpha)}{f'(\alpha)}.$$

## 4.5. **A convex combination of the Newton and a Newton-Halley type methods**

This is well known that Newton's method is given by (1) and the Newton-Halley type method [3] is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left(\frac{1}{1-BL_f(x_n)}\right).$$

Taking a convex combination of both, leads to

$$x_{n+1} = A\left(x_n - \frac{f(x_n)}{f'(x_n)}\right) + (1-A)\left(x_n - \frac{f(x_n)}{f'(x_n)}\frac{1}{1-BL_f(x_n)}\right),$$

which can be written as:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left(A + \frac{1-A}{1-BL_f(x_n)}\right).$$

So, $H(t) = A + \frac{1-A}{1-Bt} \Rightarrow H(0) = 1$ and $\dot{H}(0) = B(1-A)$. To make $\dot{H}(0) = \frac{1}{2}$, it is necessary that $B = \frac{1}{2(1-A)}$. Then the following weight function is obtained

$$H(t) = A + \frac{2(A-1)^2}{2(1-A)-t} \Rightarrow H(0) = 1, \dot{H}(0) = \tfrac{1}{2} \text{ and } \ddot{H}(0) = \frac{1}{2(1-A)}; (A \neq 1).$$

If we consider the equation

$$x = x_n + \frac{y-f(x_n)}{f'(x_n)}\left(A + \frac{2(A-1)^2}{2(1-A) - \frac{[f(x_n)-y]f''(x_n)}{\left(f'(x_n)\right)^2}}\right),$$

then by theorem 3.1, the equation of iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left(A + \frac{2(A-1)^2}{2(1-A)-L_f(x_n)}\right),$$

and

$$K(\alpha) = \frac{(1-2A)}{4(1-A)}\left[\frac{f''(\alpha)}{f'(\alpha)}\right]^2 - \frac{1}{6}\frac{f'''(\alpha)}{f'(\alpha)}; (A \neq 1).$$

If we let $A = 1$, we obtain the Newton's method (1).

This family of third-order iterative algorithms includes, as particular cases, the following methods

- $A = 0$, corresponds to Halley's method (15).
- If we let $A \to \pm\infty$, we obtain the Chebyshev's method (14).
- If $A = \frac{1}{2}$ the Super Halley's method (16) is obtained.

4.6. **A new family of Chebyshev-Halley type methods for finding simple roots of nonlinear equations**

Here we start with the $H$ weight function

$$H(t) = \frac{1}{1-At} + (\tfrac{1}{2} - A)t; \Rightarrow H(0) = 1, \dot{H}(0) = \tfrac{1}{2} \text{ and } \ddot{H}(0) = 2A^2,$$

to consider the equation

$$x = x_n + \frac{y-f(x_n)}{f'(x_n)}\left[\frac{1}{\left(1 - A\frac{[f(x_n)-y]f''(x_n)}{\left(f'(x_n)\right)^2}\right)} + (\tfrac{1}{2} - A)\frac{[f(x_n)-y]f''(x_n)}{\left(f'(x_n)\right)^2}\right],$$

which eventually leads to the equation of iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left(\frac{1}{1-AL_f(x_n)} + (\tfrac{1}{2} - A)L_f(x_n)\right), \tag{21}$$

and

$$K(\alpha) = \frac{(1-2A^2)}{2}\left[\frac{f''(\alpha)}{f'(\alpha)}\right]^2 - \frac{1}{6}\frac{f'''(\alpha)}{f'(\alpha)}; (A \neq 1).$$

The case $A = 0$ corresponds to Chebyshev's method (14), while $A = \frac{1}{2}$ corresponds to Halley's method (15). When $A = \frac{\sqrt{2}}{2}$ the following iteration equation is obtained.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left(\frac{2}{2-\sqrt{2}L_f(x_n)} + \tfrac{1}{2}(1-\sqrt{2})L_f(x_n)\right), \tag{22}$$

Similarly, when $A = -\frac{\sqrt{2}}{2}$ the following equation of iteration is obtained.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left(\frac{2}{2+\sqrt{2}L_f(x_n)} + \tfrac{1}{2}(1+\sqrt{2})L_f(x_n)\right), \tag{23}$$

with the same

$$K(\alpha) = -\frac{1}{6}\frac{f'''(\alpha)}{f'(\alpha)},$$

for both values of $A$.

### 4.7. **A simple parameter family of third-order methods for solving nonlinear equations**

Here

$$H(t) = \frac{1}{(1-At)\left[1-\left(\frac{1}{2}-A\right)t\right]} \Rightarrow H(0) = 1, \dot{H}(0) = \tfrac{1}{2} \ \mathit{and}\ \ddot{H}(0) = \frac{4A^2-2A+1}{2}.$$

Consider

$$x = x_n + \frac{y-f(x_n)}{f'(x_n)}\frac{1}{\left(1-A\frac{[f(x_n)-y]f''(x_n)}{(f'(x_n))^2}\right)\left[1-(\frac{1}{2}-A)\frac{[f(x_n)-y]f''(x_n)}{(f'(x_n))^2}\right]},$$

then the equation of iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\frac{1}{(1-AL_f(x_n))\left[1-(\frac{1}{2}-A)L_f(x_n)\right]},$$

and

$$K(\alpha) = \frac{(1+2A-4A^2)}{2}\left[\frac{f''(\alpha)}{f'(\alpha)}\right]^2 - \frac{1}{6}\frac{f'''(\alpha)}{f'(\alpha)}.$$

If $A = 0$ or $A = \frac{1}{2}$ the Halley's method (15) is obtained. In case that $A = \frac{1+\sqrt{5}}{4}$ or $A = \frac{1-\sqrt{5}}{4}$ then the following equation of iteration is obtained

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\frac{1}{\left[1-\frac{1}{2}L_f(x_n)-\frac{1}{4}\left(L_f(x_n)\right)^2\right]}, \tag{24}$$

and

$$K(\alpha) = -\frac{1}{6}\frac{f'''(\alpha)}{f'(\alpha)}.$$

4.8. **A new family mean of two Newton-Halley type methods for finding simple roots of nonlinear equations**

For
$H(t) = \frac{1}{2-At} + \frac{1}{2-(2-A)t}; \Rightarrow H(0) = 1,\ \dot{H}(0) = \frac{1}{2}\ and\ \ddot{H}(0) = \frac{1}{2}A^2 - A + 1,$
we consider the equation

$$x = x_n + \frac{y - f(x_n)}{f'(x_n)}\left(\frac{1}{2 - A\frac{[f(x_n)-y]f''(x_n)}{\left(f'(x_n)\right)^2}} + \frac{1}{2-(2-A)\frac{[f(x_n)-y]f''(x_n)}{\left(f'(x_n)\right)^2}}\right),$$

to end up with the equation of iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left(\frac{1}{2 - AL_f(x_n)} + \frac{1}{2-(2-A)L_f(x_n)}\right),$$

and

$$K(\alpha) = \frac{A(2-A)}{4}\left[\frac{f''(\alpha)}{f'(\alpha)}\right]^2 - \frac{1}{6}\frac{f'''(\alpha)}{f'(\alpha)}.$$

If $A = 0$ or $A = 2$ we have the Super-Halley's method (16), and when $A = 1$ we have the Halley's method (15).

4.9. **An *A* family**

For
$H(t) = \frac{1}{1 - \frac{1}{2}t + At^2} \Rightarrow H(0) = 1, \dot{H}(0) = \frac{1}{2}\ and\ \ddot{H}(0) = \frac{(1-4A)}{2},$
we consider

$$x = x_n + \frac{y - f(x_n)}{f'(x_n)}\frac{1}{\left[1 - \frac{1}{2}\frac{[f(x_n)-y]f''(x_n)}{\left(f'(x_n)\right)^2} + A\left(\frac{[f(x_n)-y]f''(x_n)}{\left(f'(x_n)\right)^2}\right)^2\right]},$$

to end up with the equation of iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\frac{1}{\left[1 - \frac{1}{2}L_f(x_n) + A\left(L_f(x_n)\right)^2\right]},$$

and

$$K(\alpha) = \frac{(4A+1)}{4}\left[\frac{f''(\alpha)}{f'(\alpha)}\right]^2 - \frac{1}{6}\frac{f'''(\alpha)}{f'(\alpha)}.$$

$A = 0$ corresponds to Halley's method (15), while the case of $A = -\frac{1}{4}$ yields the equation of iteration (24).

4.10. **An $A-C$ family**

$H(t) = \frac{1}{1-At} + (\frac{1}{2}-A)t + Ct^2 \Rightarrow H(0) = 1,\ \dot{H}(0) = \frac{1}{2}$ *and* $\ddot{H}(0) = 2(A^2+C)$.
with the equation

$$x = x_n + \frac{y-f(x_n)}{f'(x_n)}\left(\frac{1}{1-A\frac{[f(x_n)-y]f''(x_n)}{\left(f'(x_n)\right)^2}} + (\tfrac{1}{2}-A)\frac{[f(x_n)-y]f''(x_n)}{\left(f'(x_n)\right)^2} + C\frac{[f(x_n)-y]^2\left(f''(x_n)\right)^2}{\left(f'(x_n)\right)^4}\right),$$

lead to the equation of iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left(\frac{1}{1-AL_f(x_n)} + (\tfrac{1}{2}-A)L_f(x_n) + C\,(L_f(x_n))^2\right),$$

and

$$K(\alpha) = \frac{(1-2A^2-2C)}{2}\left[\frac{f''(\alpha)}{f'(\alpha)}\right]^2 - \frac{1}{6}\frac{f'''(\alpha)}{f'(\alpha)}.$$

If $C = 0$ a new family of Chebyshev-Halley type methods (21) is obtained. In case that $C = \frac{1-2A^2}{2}$, then the following equation of iteration is obtained.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left(\frac{1}{1-AL_f(x_n)} + (\tfrac{1}{2}-A)L_f(x_n) + \frac{1-2A^2}{2}(L_f(x_n))^2\right),$$

and

$$K(\alpha) = -\frac{1}{6}\frac{f'''(\alpha)}{f'(\alpha)}.$$

In the particular case when $A = \frac{1}{2}$ and $C = \frac{1}{4}$ a new iterative method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left(\frac{1}{1-\frac{1}{2}L_f(x_n)} + \frac{1}{4}(L_f(x_n))^2\right), \qquad (25)$$

is obtained with

$$K(\alpha) = -\frac{1}{6}\frac{f'''(\alpha)}{f'(\alpha)}.$$

4.11. **A convex combination of two members of the Newton-Halley family**

Let $H$ be the weight function with $A \neq B$ :

$H(t) = \frac{1}{2(A-B)}\left(\frac{1-2B}{(1-At)} + \frac{2A-1}{(1-Bt)}\right) \Rightarrow H(0) = 1, \dot{H}(0) = \frac{1}{2}$ *and* $\ddot{H}(0) = A+B-2AB$.
Consider

$$x = x_n + \frac{1}{2(A-B)}\frac{y-f(x_n)}{f'(x_n)}\left(\frac{1-2B}{1-A\frac{[f(x_n)-y]f''(x_n)}{\left(f'(x_n)\right)^2}} + \frac{2A-1}{1-B\frac{[f(x_n)-y]f''(x_n)}{\left(f'(x_n)\right)^2}}\right),$$

to arrive at the equation of iteration

$$x_{n+1} = x_n - \frac{1}{2(A-B)} \frac{y-f(x_n)}{f'(x_n)} \left[ \frac{1-2B}{1-A\,L_f(x_n)} + \frac{2A-1}{1-B\,L_f(x_n)} \right],$$

and

$$K(\alpha) = \frac{(1-A-B+2AB)}{2} \left[ \frac{f''(\alpha)}{f'(\alpha)} \right]^2 - \frac{1}{6} \frac{f'''(\alpha)}{f'(\alpha)}.$$

If $A = \frac{1}{2}$ or $B = \frac{1}{2}$ the Halley's method (15) is obtained. In case that $B = \frac{A-1}{2A-1}$ then the following equation of iteration is obtained

$$x_{n+1} = x_n - \frac{1}{2\left(2A^2-2A+1\right)} \frac{f(x_n)}{f'(x_n)} \left( \frac{1}{1-A\,L_f(x_n)} + \frac{(2A-1)^3}{2A-1-(A-1)\,L_f(x_n)} \right)$$

*and* $K(\alpha) = -\frac{1}{6} \frac{f'''(\alpha)}{f'(\alpha)}$.

4.12. **A bi-parametric family**

$$H(t) = 1 - \frac{1}{2A} + \frac{1}{2A(1-At)} + Bt^2 \Rightarrow H(0) = 1, \dot{H}(0) = \tfrac{1}{2} \text{ and } \ddot{H}(0) = A + 2B.$$

with the equation

$$x = x_n + \frac{1}{2(A-B)} \frac{y-f(x_n)}{f'(x_n)} \left[ 1 - \frac{1}{2A} + \frac{1}{2A\left(1 - A\,\frac{[f(x_n)-y]f''(x_n)}{\left(f'(x_n)\right)^2}\right)} + B\frac{[f(x_n)-y]^2\left(f''(x_n)\right)^2}{\left(f'(x_n)\right)^4} \right]$$

ends up with the equation of iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \left( 1 - \frac{1}{2A} + \frac{1}{2A\left(1-AL_f(x_n)\right)} + B\,(L_f(x_n))^2 \right),$$

and

$$K(\alpha) = \frac{(1-A-2B)}{2} \left[ \frac{f''(\alpha)}{f'(\alpha)} \right]^2 - \frac{1}{6} \frac{f'''(\alpha)}{f'(\alpha)}.$$

In case when $B = \frac{1-A}{2}$, then the following equation of iteration is obtained.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \left( 1 - \frac{1}{2A} + \frac{1}{2A\left(1-AL_f(x_n)\right)} + \frac{(1-A)}{2}\,(L_f(x_n))^2 \right),$$

and

$$K(\alpha) = -\frac{1}{6} \frac{f'''(\alpha)}{f'(\alpha)}.$$

4.13. **A triparametric family**

For
$H(t) = B + \frac{1-B}{1-At} + (\frac{1}{2} + A(B-1))t + Ct^2 \Rightarrow H(0) = 1, \dot{H}(0) = \frac{1}{2}$ *and*
$\ddot{H}(0) = 2(1-B)A^2 + 2C,$
with

$$x = x_n + \frac{y-f(x_n)}{f'(x_n)}\left( B + \frac{1-B}{1-A\frac{[f(x_n)-y]f''(x_n)}{\left(f'(x_n)\right)^2}} + (\tfrac{1}{2} + A(B-1))\frac{[f(x_n)-y]f''(x_n)}{\left(f'(x_n)\right)^2} + C\,\frac{[f(x_n)-y]^2\left(f''(x_n)\right)^2}{\left(f'(x_n)\right)^4} \right),$$

we have the equation of iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left(B + \frac{1-B}{1-AL_f(x_n)} + (\tfrac{1}{2} + A(B-1))L_f(x_n) + C\,(L_f(x_n))^2\right),$$

and

$$K(\alpha) = \frac{\left(1-2(1-B)A^2-2C\right)}{2}\left[\frac{f''(\alpha)}{f'(\alpha)}\right]^2 - \frac{1}{6}\frac{f'''(\alpha)}{f'(\alpha)}.$$

In the case when $C = \frac{1-2(1-B)A^2}{2}$, the following equation of iteration is obtained.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left(B + \frac{1-B}{1-AL_f(x_n)} + (\tfrac{1}{2} + A(B-1))L_f(x_n) + \frac{1-2(1-B)A^2}{2}(L_f(x_n))^2\right),$$

and

$$K(\alpha) = -\frac{1}{6}\frac{f'''(\alpha)}{f'(\alpha)}.$$

4.14. **A pentaparametric family**

The pentaparametric weight function $H$ is
$$H(t) = \frac{4+2(A+1)t+Dt^2+Et^3}{4+2At+Bt^2+Ct^3} \Rightarrow H(0) = 1, \dot{H}(0) = \frac{1}{2} \text{ and } \ddot{H}(0) = \frac{D-A-B}{2}.$$
If we consider the equation
$$x = x_n - \frac{f(x_n)}{f'(x_n)}\frac{4+2(A+1)r+Dr^2+Er^3}{4+2Ar+Br^2+Cr^3} \quad \textit{where} \quad r = \frac{[f(x_n)-y]f''(x_n)}{\left(f'(x_n)\right)^2},$$
then by theorem 3.1 the tangency conditions are satisfied. So, the equation of iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\frac{4+2(A+1)L_f(x_n)+D\left(L_f(x_n)\right)^2+E\left(L_f(x_n)\right)^3}{4+2AL_f(x_n)+B(L_f(x_n))^2+C\left(L_f(x_n)\right)^3}, \qquad (26)$$

and

$$K(\alpha) = \frac{2+A+B-D}{4}\left[\frac{f''(\alpha)}{f'(\alpha)}\right]^2 - \frac{1}{6}\frac{f'''(\alpha)}{f'(\alpha)}.$$

If $D = 2 + A + B,$ then the following equation of iteration is obtained.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\frac{4+2(A+1)L_f(x_n)+(2+A+B)\left(L_f(x_n)\right)^2+E\left(L_f(x_n)\right)^3}{4+2AL_f(x_n)+B(L_f(x_n))^2+C\left(L_f(x_n)\right)^3},$$

and

$$K(\alpha) = -\frac{1}{6}\frac{f'''(\alpha)}{f'(\alpha)}.$$

With the exception of the Hansen-Patrick's family (10) the other families presented in this section can be generated from (26).

## 5. Numerical Examples

In order to assess the advantages and precision of the numerical schemes to solve some nonlinear scalar equations, we have applied several methods to six different examples. A comparisons made with results of some classical methods with only a few of the methods reported in this work. The methods compared are the following: NM: Newton's method (1), ChM: Chebyshev's method (14), HM: Halley's method (15), SHM: Super-Halley's method (16), OM: Ostrowski's method (11), EM: Euler's method (12) and five methods presented in this paper: CM1: equation (20), CM2: equation (22), CM3: equation (23), CM4: equation (24), and CM5: equation (25).

Here are the test functions and the approximations ($x^*$) to the root $\alpha$ to be calculated.

$f_1(x) = e^{x^2+7x-30} - 1$ $\quad x^* = 3,$

$f_2(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5$ $\quad x^* = -1.207647827130918927009416758356 08\ldots,$

$f_3(x) = e^x - 4x^2$ $\quad x^* = 0.714805912362777806137622208111809\ldots,$

$f_4(x) = x^5 + x^4 + 4x^2 - 15$ $\quad x^* = 1.347428098968304981506715380714821\ldots,$

$f_5(x) = (x-1)^6 - 1$ $\quad x^* = 2,$

$f_6(x) = x^5 - 10$ $\quad x^* = 1.584893192461113485202101373391507\ldots,$

These test functions are taken from [17].

In order to evaluate the accuracy of the numerical schemes, the computational order of convergence ($COC$) is computed, which is given, [26], as

$$COC = \frac{\ln\left|(x_{n+1}-x^*)/(x_n-x^*)\right|}{\ln\left|(x_n-x^*)/(x_{n-1}-x^*)\right|}.$$

Table 1 summarizes a comparison of various iterative methods under the same total number of function evaluations (TNFE=12). Thus $|x_n - x^*|$ is an approximation to the absolute error, where $n = 4$ in all methods apart from in Newton's method where it is six; $|f_i(x_n)|, i = 1,2,3,4,5,6$ is the absolute value of the test function $f_i$ in $x_n$, *Digits* is the number of digits used in Maple to perform the calculations except when the Ostrowski's method is applied to the test function $f_6$ ($Digits = 300$). The computational order of convergence ($COC$) is also listed in this table. Representative values of a very good, or very bad, behavior are highlighted in bold. In the case of the test function $f_1$, $x_0 = 3.2$. It can be seen that the Super-Halley's method diverges, the Euler's method converges initially very slowly and CM2 is the method with the better behavior. All other methods have an acceptable behavior. In the case of the test function $f_2$ , $x_0 = -1.4$. Clearly here, Euler's method converges initially very slowly and the Halley's and CM3 methods exhibit a better behavior. All other methods have a reasonably good behavior.

Table 1: Comparison of solutions by various iterative methods under the same total number of function evaluations (TNFE=12).

| | $\lvert x_n - x^*\rvert$ | $\lvert f_1(x_n)\rvert$ | *COC* | $\lvert x_n - x^*\rvert$ | $\lvert f_2(x_n)\rvert$ | *COC* |
|---|---|---|---|---|---|---|
| | $f_1(x),\ x_0 = 3.2$ | $Digits = 50$ | | $f_2(x),\ x_0 = -1.4$ | $Digits = 90$ | |
| NM | $3.4458738e-8$ | $4.4796369e-7$ | 1.9963578 | 1.0034065e-36 | 2.0376588e-35 | 2.0000000 |
| ChM | $1.1164122e-8$ | $1.4513359e-7$ | 2.9423348 | 1.9925228e-44 | 4.0463080e-43 | 2.9999998 |
| HM | $2.5127358e-17$ | $3.2665565e-16$ | 3.0000839 | **1.2674286e-74** | **2.5738193e-73** | **3.0000000** |
| SHM | **21.871192** | **1.6801564e-331** | **0.8004937** | 1.7487793e-42 | 3.5513182e-41 | 3.0000013 |
| OM | $6.5067815e-19$ | $8.4588159e-18$ | 3.2812794 | 6.282255e-58 | 1.3257136e-56 | 3.0000000 |
| EM | **5.3834699e-4** | **6.9754082e-3** | **4.7082703** | **6.9956373e-26** | **1.4206329e-24** | **3.0002348** |
| CM1 | $8.9449288e-23$ | $1.1628407e-21$ | 3.0013841 | 9.6433837e-53 | 1.9583216e-51 | 2.9999999 |
| CM2 | **8.9301538e-40** | **1.1609200e-38** | **3.0000058** | 2.3595619e-48 | 4.7916595e-47 | 3.0000000 |
| CM3 | $7.8614527e-15$ | $1.0219889e-13$ | 2.9248581 | **3.3509768e-68** | **6.8049664e-67** | **3.0000000** |
| CM4 | $9.7186510e-25$ | $1.2634246e-23$ | 2.9996207 | 6.5716416e-47 | 1.3345303e-45 | 3.0000000 |
| CM5 | $1.3953424e-25$ | $1.8139451e-24$ | 3.0009017 | 6.6689010e-58 | 1.3542812e-56 | 3.0000000 |

| | $\lvert x_n - x^*\rvert$ | $\lvert f_3(x_n)\rvert$ | *COC* | $\lvert x_n - x^*\rvert$ | $\lvert f_4(x_n)\rvert$ | *COC* |
|---|---|---|---|---|---|---|
| | $f_3(x),\ x_0 = 0.4$ | $Digits = 90$ | | $f_4(x),\ x_0 = 1.0$ | $Digits = 90$ | |
| NM | 2.6648543e-29 | 9.7924264e-29 | 2.0000000 | 8.2298858e-25 | 3.0488534e-23 | 1.9999999 |
| ChM | **1.2112256e-5** | **4.4507952e-5** | **3.3624416** | 3.5682240e-19 | 1.3218885e-17 | 3.0015628 |
| HM | 6.7049483e-35 | 2.4638387e-34 | 3.0000273 | 7.0408823e-50 | 2.6083737e-48 | 3.0000001 |
| SHM | 2.8201153e-59 | 1.0362957e-58 | 3.0000002 | 1.1374924e-42 | 4.2139681e-41 | 3.0000011 |
| OM | 3.0785197e-50 | 1.1312505e-49 | 3.0000003 | **5.5000059e-76** | **2.0375388e-74** | **3.0000000** |
| EM | **1.0819900e-85** | **3.9760000e-85** | **2.9999999** | 4.0491220e-48 | 1.5000426e-46 | 2.9999997 |
| CM1 | 1.4846884e-31 | 5.4557209e-31 | 3.0011856 | 9.2106967e-32 | 3.4122057-30 | 3.0001277 |
| CM2 | 1.4168049e-39 | 5.2062727e-39 | 3.0000909 | 3.0177405e-35 | 1.1179558e-33 | 3.0001277 |
| CM3 | **1.1225727** | **3.5761338e-7** | **0.0434761** | **0.1017240** | **4.2013589** | **3.0000314** |
| CM4 | 5.7559417e-33 | 2.1151113e-32 | 3.0005436 | 1.3231360e-33 | 4.9017057e-32 | 2.0381848 |
| CM5 | 1.6194951e-21 | 5.9510897e-21 | 3.0230848 | 1.8459608e-29 | 6.8385684e-28 | 3.0003483 |

For the test function $f_3,\ x_0 = 0.4$, and it can be seen that Chebyshev's method converges initially very slowly, CM3 methods converges initially very slowly to other root ($x^* = -0.40777670940448032888636366265427974029870495442031926994 68\cdots$) and Euler's method exhibits a better behavior. All other methods have an acceptable behavior while highlighting the Super Halley's method.

For the test function $f_4,\ x_0 = 1$; it can be seen that the CM3 method converges initially extremely slow but Ostrowski's method shows a better behavior; all other methods have an acceptable behavior while highlighting the very slow convergence of the Chebyshev's method. In the case of the test function $f_5,\ x_0 = 1.8$; clearly Chebyshev's method diverges, the CM3 method converges initially very slowly and the Ostrowski's method exhibits a better behavior; all other methods have an acceptable behavior highlighting the slow convergence of the CM5 method. Finally, for the test function $f_6,\ x_0 = 1.4$, Chebyshev's method converges initially slowly and Ostrowski's method produces a better behavior (behaves like a method of fourth order). Here also all other methods have an acceptable behavior. It is noteworthy that in all of these tests, Newton's method has exhibited a good behavior.

Table 1: Comparison of solutions by various iterative methods under the same total number of function evaluations (TNFE=12); continuation.

| | $\lvert x_n - x^*\rvert$ | $\lvert f_5(x_n)\rvert$ | $COC$ | $\lvert x_n - x^*\rvert$ | $\lvert f_6(x_n)\rvert$ | $COC$ |
|---|---|---|---|---|---|---|
| | $f_5(x),\ x_0 = 1.8$ | $Digits = 90$ | | $f_6(x),\ x_0 = 1.4$ | $Digits = 300$ | |
| NM | 1.6215643e-15 | 9.7293858e-15 | 1.9999716 | 3.2578482e-38 | 1.0277816e-36 | 2.0000000 |
| ChM | 6.9986323e11 | 1.1751115e71 | -0.9800589e-2 | 7.2524521e-38 | 2.2879940e-36 | 3.0000076 |
| HM | 9.1431097e-37 | 5.4858658e-36 | 3.0000050 | 2.5366069e-62 | 8.0024538e-61 | 3.0000000 |
| SHM | 4.7722758e-31 | 2.8633655e-30 | 3.0000397 | 9.8433588e-61 | 3.1053698e-59 | 3.0000000 |
| OM | 1.7564404e-71 | 1.0538642e-70 | 2.7492540 | 9.5635873e-236 | 3.0171078e-234 | 4.0000000 |
| EM | 1.2304430e-38 | 7.3826580e-38 | 2.9999958 | 2.1586167e-65 | 6.8099753e-64 | 3.0000000 |
| CM1 | 7.7954209e-22 | 4.6772525e-21 | 2.9970332 | 2.0932788e-51 | 6.6038483-50 | 3.0000001 |
| CM2 | 1.0520471e-22 | 6.3122824e-22 | 3.0015009 | 8.3223813e-55 | 2.6255338e-53 | 3.0000002 |
| CM3 | 2.1792112e-5 | 1.3074554e-4 | 3.4928718 | 3.4222357e-46 | 1.0796708e44 | 3.0000021 |
| CM4 | 6.1360087e-18 | 3.6816052e-17 | 3.0058168 | 1.2671070e-54 | 3.9974522e-53 | 3.0000002 |
| CM5 | 4.0788264e-11 | 2.4472958e-10 | 2.5854025 | 2.7793769e-47 | 8.7683412e-46 | 3.0000029 |

## 6. Conclusions

In this paper, the extended Gander's theorem and an outcome related to geometric interpretation of the iterative methods obtained therefrom are presented. The procedure summarize as: for any weight $H$ function that meets the hypothesis Gander's theorem, it is possible to obtain immediately the asymptotic error constant together with a pertaining geometric interpretation. Different families and methods, both new and classic, are presented.

It is clear that a further study is needed for the new families derived in this paper. Particularly for those related to the dynamic behavior in the sphere of Riemann. A study, similar to the one conducted in this work, can also be performed for different families and / or methods, such as those reported in [6].

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## References

[1] S. Amat, S. Busquier, and J. M. Gutiérrez,Geometric constructions of iterative functions to solve nonlinear equations, *Journal of Computational and Applied Mathematics* **157**, (2003), 197-205.

[2] S. Amat, S. Busquier, and S. Plaza, On the dynamics of a family of third-order iterative functions, *ANZIAM Journal* **48**, (2007), 343-359.

[3] C. E. Cadenas, A family of Newton-Halley type methods to find simple roots of nonlinear equations and their dynamics. Accepted in 2017 for publication in *Communications in Numerical Analysis*.

[4] B. Campos, A. Cordero, Á. A. Magreñan, J. R. Torregrosa, and P. Vindel, Study of a biparametric family of iterative methods, *Abstract and Applied Analysis* **2014**, (2014), ID 141643, 12 pages.

[5] B. Campos, A. Cordero, J. R. Torregrosa, and P. Vindel, Dynamics of the family of c-iterative methods, *International Journal of Computer Mathematics*, **92**(9), (2015), 1815-1825.

[6] B. Campos, A. Cordero, J. R. Torregrosa, and P. Vindel, Behaviour of fixed and critical points of the $(\alpha, c)$-family of iterative methods, *Journal of Mathematical Chemistry*, **53**, (2015), 807–827.

[7] A. Cordero, J. R. Torregrosa, and P. Vindel, Dynamics of a family of Chebyshev-Halley type methods, *Applied Mathematics and Computation* **219** (16), (2013), 8568-8583.

[8] A. Cordero, C. Jordan, and J. R. Torregrosa, One-point Newton-type iterative methods: A unified point of view, *Journal of Computional and Applied Mathematics* **275**, (2015), 366-374.

[9] J. A. Ezquerro, J. M. Gutiérrez, , M. A. Hernández, and M. A. Salanova, Chebyshev-like methods and quadratic equations, *Revue D'Analyse Numérique et de Théorie de L'Approximation* **28** (1), (1999), 23-35.

[10] W. Gander, On Halley's iteration method, *American Mathematical Monthly* **92**, (1985), 131-134.

[11] J. M. Gutiérrez, and M. A. Hernández, A family of Chebyshev-Halley type methods in Banach spaces, *Bulletin of the Australian Mathematical Society* **55**, (1997), 113-130.

[12] J. M. Gutiérrez, and M. A. Hernández, An acceleration of Newton's method: Super-Halley method, *Applied Mathematics and Computation* **117**(2), (2001), 223-239.

[13] E. Halley, Methodus nova accurata et facilis inveniendi radices æquationum quarumcumque generaliter, *Sine Praevia Reductione*, *Philosophical Transactions of the Royal Society A* **18**, (1694), 136-148.

[14] E. Hansen, and M. Patrick, A family of root finding method, *Numerical Mathematics* **27**, (1977), 257-269.

[15] M. A. Hernández, Newton-Raphson's method and convexity, *Review of Research, Faculty of Science, Mathematics Series (Zb. Rad. Prirod.-Mat. Fak. Ser. Mat.)* **22** (1), (1992), 159-166.

[16] V. Kanwar, S. Singh, and S. Bakshi, Simple geometric constructions of quadratically and cubically convergent iterative functions to solve nonlinear equations, *Numerical Algorithms* **47**, (2008), 95-107.

[17] L. Liu, and X. Wang, Eighth-order methods with haigh efficiency index for solving nonlinear equations, *Applied Mathematics and Computation* **215**, (2010), 3449-3454.

[18] A. Melman, Geometry and convergence of Euler's and Halley's methods, *SIAM Review* **39**(4), (1997), 728-735.

[19] B. Neta, Several new methods for solving equations, *Numerische Mathematik* **23**(3-4), (1988), 265-282.

[20] M. G. Olivo, and J. M. Gutiérrez, Notas históricas sobre el método de Chebyshev para resolver ecuaciones no lineales, *Miscelánea Matemática* **57**, (2013), 63-83.

[21] M. G. Olivo, *El Método de Chebyshev Para el Cálculo de las Raices Ecuaciones no Lineales*, Doctoral Dissertation, Universidad de La Roja, 2013.

[22] A. M. Ostrowski, *Solution of Equations and Systems of Equations*, Prentice-Hall, Engle-Wood Cliffs, NJ, 1964.

[23] T. R. Scavo, and J. B. Thoo, On the geometry of Halley's method, *American Mathematical Monthly* **102**, (1995), 417-426.

[24] E. Schröder. Über unendlichviele algorithm zur auffosung der gleichungen, *Mathematische Annalen* **2**, (1870), 317-365.

[25] J. F. Traub, *Iterative Methods for Resolution of Equations*, Prectice Hall, NJ, 1964.

[26] S. Weerakoon, and T. G. I. Fernando, A variant of Newton's method with accelerated third-order convergence, *Applied Mathematics Letters* **13**(8), (2000), 87–93.

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